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HOMOGENIZATION OF FLOW IN DOUBLE-POROUS SOLIDS WITH APPLICATIONS IN BIOMECHANICS

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Summary: *The paper summarizes recent efforts in the homogenization based multiscale modelling of biological tissues in the context of the fluid-structure interaction at the microscopic level. Two models are presented: a model of compact bone poroelasticity and a model of parallel flows in perfused deforming tissue. The homogenization approach employed to develop these models is based on the Biot model of fluid saturated porous media, assumptions of periodic geometry of the pores and uses scale dependent permeability in the double-porous compartments.*

1. Introduction

Biological tissues are multiphase media which, in general, are constituted by solid skeleton and interstitial fluids. This rough modelling scheme applies for both hard and soft tissues, although in many biomechanical studies the tissue is considered just as a (visco)elastic solid – such an approximation is relevant if the task is to provide a stress-strain analysis as a response to short-period events, dynamic loading in crash tests, or static analyses, when the complex interactions and the multi-physic character of processes in living tissues can be neglected. Nowdays, however, the biomechanical and biomedical research is more focused on the tissue growth, remodelling and biochemical processes which are closely related to the biological fluid transport and mechanical fluctuations of stresses and strains at the cellular level. In the paper we present some biomechanical models which all are developed using homogenization of fluid saturated porous solid, where the flow is described by the Darcy law. The so-called strong heterogeneities (discontinuities) in the hydraulic permeability, K, are considered, so that in some parts of the microstructure which corresponds to the dual porosity, K depends on ε^2, where ε is the length of the heterogeneity period. The strong heterogeneity results in a new quality of the macroscopic constitutive laws in comparison with those defined at the microscopic level.

The aim of the paper is to report recent investigations concerning homogenization of the Biot model of the fluid saturated porous media (FSPM). Using the "ε-square scaling" of the permeability coefficients we account for the dual porosity of the medium matrix. Two topological types of the microstructure are considered:

- *one connected channel system* – model of compact bone poroelastic properties; this

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present a quite new result, details on the model development will be published in a forthcoming paper. A similar model with different topology of the microstructure (dual-porous inclusions in a porous matrix) was treated in Griso & Rohan (2007).

- *two (or more) disjoint channel systems* – model of deforming perfused tissue, where the channel systems reflect the arterial and venous sectors; these are mutually separated each other by the dual porous matrix representing the tissue penetrated by a very fine network of arterioles and capillaries. This topic was discussed e.g. in Rohan et al. (2007), a similar topology of the strongly heterogeneous medium was treated in Showalter & Visarraga (2004) in a much simpler application to parallel heat diffusion.

In Sections 2. and 3. we explain *the structure* of the homogenized models, i.e. the local problems for the so-called corrector functions and the global problems which allow to compute the macroscopic state (displacement and pressure fields). While the local problems resemble structure of the "original micromodel" represented by the Biot model, the "upscaled" (global) models involving the homogenized coefficients are different in their structure, exhibiting the *fading memory* effects.

Models of heterogeneous material are expressed in terms of quantities depending on the (global) "macro"- and (local) "micro"-scale coordinates x and y, respectively; they are related each other by relation $x = \varepsilon y$, as customary, see e.g. Hornung (1997); Cioranescu & Donato (1999) for the general setting of the homogenization technique and Cioranescu et al. (2002) for basics of the periodic unfolding technique employed also to develop the models reported in this paper.

2. Poroelastic properties of compact bone

We consider heterogeneities at the following levels:

1. Compact bone level formed by the *osteons*. The upscaled model is relevant to the "macroscopic" parts of the compact bone considered as a homogeneous medium, but it inherits structural (thereby the mechanical) properties of the underlying levels.

2. At the *osteon level* (of the scale ε) we consider distinguishable heterogeneities, as represented by the Haversian and Volkmann *channels* embedded in the "matrix".

3. The *matrix* contains canaliculi, another (dual) porous structure which is *not distinguishable* in the sense of subdomains of positive volume measures. Canaliculi are drained in the *channels* of the "upper" porosity level. In Fig. 1, the dual porous structure of the osteon is displayed schematically, where the second scale $\varepsilon\delta$ is apparent

The details on homogenization of the heterogeneous *periodic* structure specified above will be given in a forthcoming publication. Here we shall present just the limit model obtained by asymptotic analysis of the following problem. We define the representative periodic cell (RPC) $Y = \Pi_{i=1}^{3}]0, \bar{y}_i[$, $\mathbb{R} \ni \bar{y}_i > 0$ which generates the periodic structure of the characteristic size ε. Let Y_c be a (connected) subdomain of Y with Lipschitz boundary representing the Haverse–

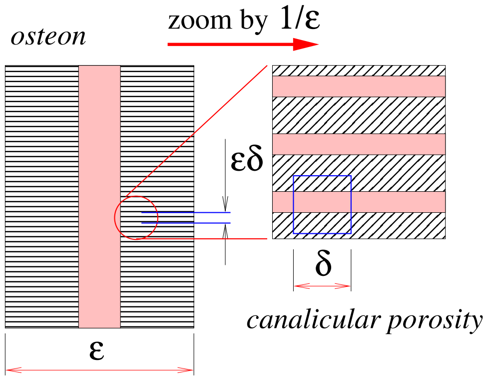

Figure 1: Dual porous structure of the osteon (cylinder) whose diameter is $\approx \varepsilon$. Haverse canal in pink (left) and an idealized structure of canalicular porosity (right) with its characteristic scale $\approx \varepsilon\delta$

Volkmann *channels*, see Fig. (2),

$$\begin{aligned} Y_m &= Y \setminus \overline{Y_c} \,, \\ \partial_m Y_c = \partial_c Y_m &= \overline{Y_c} \cap \overline{Y_m} \,, \\ \partial Y_c \cap \partial Y &\neq \emptyset \,, \end{aligned} \tag{1}$$

where Y_m is the *matrix* compartment. While in Y_c the permeability (related to the Darcy flow) is *independent of the scale*, ε, in Y_m due to the dual porosity $K^\varepsilon_{ij}(y) \approx \varepsilon^2$, $y \in Y_m$.

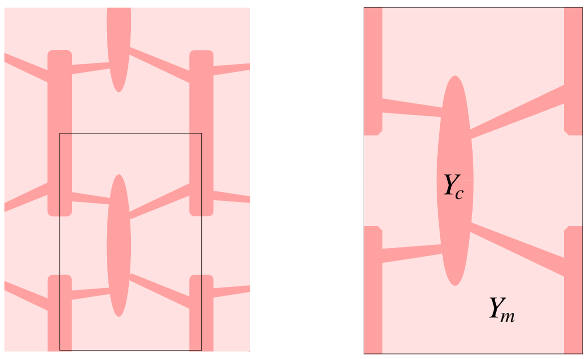

Figure 2: Microstructure of the compact bone: Left: Haverse–Volkmann system of interconnected channels (dark pink) and the matrix (light pink). Right: the RPC, Y, generating the periodic structure; decomposition (1) of Y.

The diffusion-deformation problem with finite scale heterogeneities reads as: for a.a. $t \in]0, T[$ find $\boldsymbol{u}^\varepsilon(t) \in V$ and $p^\varepsilon(t) \in H^1(\Omega)$ such that in the sense of time distributions

$$\begin{aligned} \int_\Omega D^\varepsilon_{ijkl} e_{kl}(\boldsymbol{u}^\varepsilon) e_{ij}(\boldsymbol{v}) - \int_\Omega p^\varepsilon \, \alpha^\varepsilon_{ij} e_{ij}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,, \quad \forall \boldsymbol{v} \in V_0 \,, \\ \int_\Omega q \, \alpha^\varepsilon_{ij} e_{ij}(\frac{\mathrm{d}}{\mathrm{d}\, t} \boldsymbol{u}^\varepsilon) + \int_\Omega K^\varepsilon_{ij} \partial_j p^\varepsilon \, \partial_i q + \int_\Omega \frac{1}{\mu^\varepsilon} \frac{\mathrm{d}}{\mathrm{d}\, t} p^\varepsilon \, q = 0 \,, \quad \forall q \in H^1(\Omega) \,, \end{aligned} \tag{2}$$

where $\boldsymbol{u}(0,x) = 0$ and $p(0,x) = 0$ for a.a. $x \in \Omega$. (We require $\bar{\boldsymbol{u}}(0,\cdot) = 0$ on $\partial\Omega$.) Above the sets of admissible displacements are employed:

$$
\begin{aligned}
V &= \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) \,|\, \boldsymbol{v} = \bar{\boldsymbol{u}}(t,\cdot) \text{ on } \partial\Omega,\ t \in]0,T[\}\ , \\
V_0 &= \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) \,|\, \boldsymbol{v} = 0 \text{ on } \partial\Omega\}\ .
\end{aligned}
\tag{3}
$$

In (2), D^{ε}_{ijkl} is the elastic tensor, $\alpha^{\varepsilon}_{ij}$ are the Biot coefficients and μ^{ε} is the Biot modulus.

2.1. Microscopic local problems for the corrector basis functions

In the sequel, to simplify the notation, the following bilinear forms are used

$$
\begin{aligned}
a_Y\left(\boldsymbol{u},\, \boldsymbol{v}\right) &= \fint_Y D_{ijkl}(y) e^y_{kl}(\boldsymbol{u})\, e^y_{ij}(\boldsymbol{v})\ , \\
b_Y\left(\varphi,\, \boldsymbol{v}\right) &= \fint_Y \varphi\, \alpha_{ij}(y) e^y_{ij}(\boldsymbol{v})\ , \\
b_{Y_m}\left(\varphi,\, \boldsymbol{v}\right) &= \fint_{Y_m} \varphi\, \alpha^m_{ij}(y) e^y_{ij}(\boldsymbol{v})\ , \\
c_{Y_m}\left(\varphi,\, \psi\right) &= \fint_{Y_m} K^m_{ij}(y) \partial^y_j \varphi \partial^y_i \psi\ , \\
d_{Y_m}\left(\varphi,\, \psi\right) &= \fint_{Y_m} (\mu^m)^{-1} \psi\, \varphi\ .
\end{aligned}
\tag{4}
$$

Further one introduces $\boldsymbol{\Pi}^{rs} = (\Pi^{rs}_i)$, where $\Pi^{rs}_i = y_s \delta_{ir}$. The following spaces are employed: $\mathbf{H}^1_\#(Y)$ denotes the restriction of the Sobolev space $\mathbf{H}^1(Y)$ to the Y-periodic vectorial functions, whereby analogous notation $H^1_\#(Y)$ is used for scalar functions; for $Z \subset Y$, $H^1_{\#0}(Z)$ is the restriction of $H^1_\#(Z)$ to functions which are zero on $\partial Z \setminus \partial Y$ (i.e. periodic on $\partial Y \cap \partial Z$).

The limit homogenized model derived from (2) involves macroscopic displacements, $\boldsymbol{u}$ and pressure, p, which is associated with (macroscopic) flow in the Haverse-Volkmann porosity. The macroscopic model involves homogenized parameters which depend on characteristic response of the microstructure reflecting essentially the phenomenon of coupled deformation and microflow. These characteristic responses are expressed in terms of the corrector basis functions.

Steady problem for strain-associated correctors. Couple $(\bar{\boldsymbol{\omega}}^{rs}, \bar{\pi}^{rs}) \in \mathbf{H}^1_\#(Y) \times H^1_{\#0}(Y_m)$ is the solution of

$$
\begin{aligned}
a_Y\left(\bar{\boldsymbol{\omega}}^{rs},\, \boldsymbol{v}\right) &= -a_Y\left(\boldsymbol{\Pi}^{rs},\, \boldsymbol{v}\right) \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\#(Y)\ , \\
c_{Y_m}\left(\bar{\pi}^{rs},\, q\right) &= -b_{Y_m}\left(q,\, \bar{\boldsymbol{\omega}}^{rs} + \boldsymbol{\Pi}^{rs}\right) \quad \forall q \in H^1_{\#0}(Y_m)\ .
\end{aligned}
\tag{5}
$$

Steady problem for pressure-associated correctors. Couple $(\boldsymbol{\omega}^{*,P}, \tilde{\pi}^P(0_+)) \in \mathbf{H}^1_\#(Y) \times H^1_{\#0}(Y_m)$ is the solution of

$$
\begin{aligned}
a_Y\left(\boldsymbol{\omega}^{*,P},\, \boldsymbol{v}\right) - b_{Y_m}\left(\tilde{\pi}^P(0_+),\, \boldsymbol{v}\right) &= b_Y\left(1,\, \boldsymbol{v}\right) \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\#(Y)\ , \\
b_{Y_m}\left(q,\, \boldsymbol{\omega}^{*,P}\right) + d_{Y_m}\left(\tilde{\pi}^P(0_+),\, q\right) &= -d_{Y_m}\left(1,\, q\right) \quad \forall q \in H^1_{\#0}(Y_m)\ .
\end{aligned}
\tag{6}
$$

Generic evolutionary problem for correctors. We define couple $(\tilde{\boldsymbol{\omega}}(t), \tilde{\pi}(t)) \in \mathbf{H}^1_\#(Y) \times H^1_{\#0}(Y_m)$ as the solution of the following evolutionary system

$$
\begin{aligned}
a_Y\left(\tilde{\boldsymbol{\omega}},\, \boldsymbol{v}\right) - b_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\pi},\, \boldsymbol{v}\right) &= 0 \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\#(Y)\;, \\
b_{Y_m}\left(q,\, \boldsymbol{\omega}\right) + c_{Y_m}\left(\tilde{\pi},\, q\right) + d_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\pi},\, q\right) &= 0 \quad \forall q \in H^1_{\#0}(Y_m)\;,
\end{aligned} \tag{7}
$$

where $\tilde{\pi}(0)$ is defined according to the following types of corrector functions:

- ∘ strain-associated correctors $(\tilde{\boldsymbol{\omega}}^{rs}, \tilde{\pi}^{rs})$ solving (7) for $\tilde{\boldsymbol{\omega}}^{rs} \equiv \tilde{\boldsymbol{\omega}}$, $\tilde{\pi}^{rs} \equiv \tilde{\pi}$, $\tilde{\pi}^{rs}(0) := -\bar{\pi}^{rs}$;
- ∘ pressure-associated correctors $(\tilde{\boldsymbol{\omega}}^{\alpha}, \tilde{\pi}^{\alpha})$ solving (7) for $\tilde{\boldsymbol{\omega}}^{\alpha} \equiv \tilde{\boldsymbol{\omega}}$, $\tilde{\pi}^{\alpha} \equiv \tilde{\pi}$, $\tilde{\pi}^{\alpha}(0)$ obtained by (6).

Effective permeability $\mathcal{C}_{ij}$ relevant to the macroscopic scale is computed using corrector basis functions $\eta^k \in H^1_\#(Y)/\mathbb{R}$, $k = 1, 2, 3$ which satisfy the following *autonomous auxiliary problem* imposed in channels Y_c:

$$
\int\!\!\!\!\!\!-_{Y_c} K^c_{ij} \partial^y_j (\eta^k + y_k)\, \partial^y_i \psi = 0 \quad \forall \psi \in H^1_\#(Y)\;. \tag{8}
$$

Using correctors η^k the *homogenized permeability* associated with the Darcy flow in the channels is computed:

$$
\mathcal{C}_{kl} = \int\!\!\!\!\!\!-_{Y_c} K^c_{ij} \partial^y_j (\eta^l + y_l)\, \partial^y_i (\eta^k + y_k)\;. \tag{9}
$$

2.2. Macroscopic model, homogenized coefficients

The macroscopic (homogenized) model involves the following homogenized material parameters which are evaluated using the characteristic responses obtained on solving (5)-(7)

- the homogenized *elastic tensor*

$$
\mathcal{E}_{ijkl} = a_Y\left(\boldsymbol{\Pi}^{kl} + \bar{\boldsymbol{\omega}}^{kl},\, \boldsymbol{\Pi}^{ij} + \bar{\boldsymbol{\omega}}^{ij}\right)\;, \tag{10}
$$

- the homogenized *viscosity tensor* of the fading memory

$$
\mathcal{H}_{ijkl}(t) = c_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\pi}^{kl},\, \bar{\pi}^{ij}\right)\;, \tag{11}
$$

- the “elastic” homogenized *Biot coefficients*

$$
\mathcal{B}_{ij} = \int\!\!\!\!\!\!-_{Y} \alpha_{ij} + b_Y\left(1,\, \bar{\boldsymbol{\omega}}^{ij}\right)\;, \tag{12}
$$

- the "fading memory" homogenized *Biot coefficients*

$$\mathcal{F}_{ij}(t) = b_Y\left(1,\, \tilde{\boldsymbol{\omega}}^{ij}(t)\right) + d_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^{ij},\, 1\right)\ , \tag{13}$$

- the homogenized *reciprocal Biot modulus* (instantaneous response)

$$\mathcal{M} = \fint_Y \frac{1}{\mu} + d_{Y_m}\left(\tilde{\pi}^P,\, 1\right) + b_Y\left(1,\, \boldsymbol{\omega}^{*,P}\right)\ , \tag{14}$$

- the fading memory effect of the homogenized *reciprocal Biot modulus*

$$\mathcal{G} = d_{Y_m}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^P,\, 1\right) + b_Y\left(1,\, \tilde{\boldsymbol{\omega}}^P\right)\ . \tag{15}$$

Macroscopic problem — coupled diffusion-deformation in the compact bone. For a.a. $t \in]0, T[$ find $\boldsymbol{u} \in V$ and $p \in H^1(\Omega)$ with $p(0) = 0$ (we assumed unloaded stress-free initial structure) such that

$$\begin{aligned}
&\int_\Omega \mathcal{E}_{ijkl} e_{kl}(\boldsymbol{u}) e_{ij}(\boldsymbol{v}) + \int_\Omega \int_0^t \mathcal{H}_{ijkl}(t-\tau) e_{kl}(\frac{\mathrm{d}}{\mathrm{d}\,\tau}\boldsymbol{u}(\tau))\, d\tau\, e_{ij}(\boldsymbol{v}) \\
&- \int_\Omega (\mathcal{B}_{ij} + \mathcal{F}_{ij}(0_+))\, p\, e_{ij}(\boldsymbol{v}) - \int_\Omega \int_0^t \mathcal{F}_{ij}(t-\tau) p(\tau)\, d\tau\, e_{ij}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\ ,
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega \mathcal{B}_{ij} e_{ij}(\boldsymbol{u})\, q + \int_\Omega \mathcal{F}_{ij}(0_+) e_{ij}(\frac{\mathrm{d}}{\mathrm{d}\,t}\boldsymbol{u})\, q + \int_\Omega \int_0^t \frac{\mathrm{d}}{\mathrm{d}\,t}\mathcal{F}_{ij}(t-\tau) e_{ij}(\frac{\mathrm{d}}{\mathrm{d}\,\tau}\boldsymbol{u}(\tau))\, d\tau\, q \\
&+ \int_\Omega \mathcal{C}_{ij} \partial_j p \partial_i q + \int_\Omega \left(\mathcal{M}\frac{\mathrm{d}}{\mathrm{d}\,t}p + \mathcal{G}(0_+) p\right) q + \int_\Omega \int_0^t \mathcal{G}(t-\tau) p(\tau)\, d\tau\, q = 0\ ,
\end{aligned} \tag{16}$$

for all $\boldsymbol{v} \in V_0$ and $q \in H^1(\Omega)$.

It is worth noting that:

- Apparent viscoelastic behaviour inherited from the microflow effects is invoked in all homogenized forms of the original Biot model coefficients.
- System (16) is symmetric (due to the Biot coefficients).
- The effective Biot moduli are associated not only to $\frac{\mathrm{d}}{\mathrm{d}\,t}p$, but also to the instantaneous pressure, p, and to the fading memory effects.

3. Perfusion of deforming tissue

In this section we report a model similar to that of the bone tissue introduced in Section 2.. Now we consider *two systems of channels* separated by the matrix interface. Such a model is applicable to describe the diffusion-deformation phenomena related to the blood perfusion in deforming tissue.

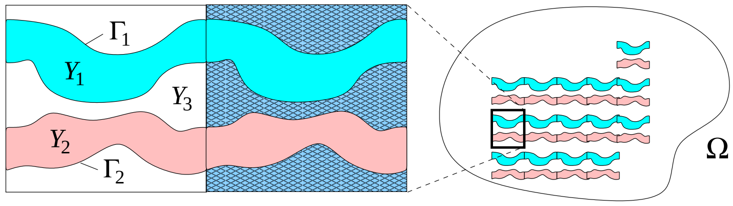

Figure 3: Left: the reference microstructural cell split in 2 high conductive sectors Y_1 and Y_2 separated by interface (matrix) sector Y_3; $\Gamma_k = \overline{Y_k \cap Y_3}$, $k = 1, 2$. Right: the periodic lattice structure of domain Ω.

We assume the medium is generated as the periodic lattice by the reference microstructural cell Y which is split into three non-overlapping sectors Y_k, $k = 1, 2, 3$, so that $\overline{Y} = \overline{Y_1 \cup Y_2 \cup Y_3}$ with interfaces $\Gamma_k = \overline{Y_k \cap Y_3}$, see Fig. 3.

In analogy with (2), the following weak formulation is considered for homogenization: for a.a. $t \in]0, T[$ find $\boldsymbol{u}^\varepsilon(t) \in V$ and $p^\varepsilon(t) \in H^1(\Omega)$ such that in the sense of time distributions

$$
\begin{aligned}
\int_\Omega D^\varepsilon_{ijkl} e_{kl}(\boldsymbol{u}^\varepsilon) e_{ij}(\boldsymbol{v}) - \int_\Omega p^\varepsilon \,\mathrm{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \;, \quad \forall \boldsymbol{v} \in V_0 \;, \\
\int_\Omega q \,\mathrm{div} \frac{\mathrm{d}}{\mathrm{d}\, t} \boldsymbol{u}^\varepsilon + \int_\Omega K^\varepsilon_{ij} \partial_j p^\varepsilon \,\partial_i q = 0 \;, \quad \forall q \in H^1(\Omega) \;,
\end{aligned}
\tag{17}
$$

where $\boldsymbol{u}(0, x) = 0$ and $p(0, x) = 0$ for a.a. $x \in \Omega$. (We require $\bar{\boldsymbol{u}}(0, \cdot) = 0$ on $\partial\Omega$.)

Above the constitutive parameters vary with position in the microstructure, being Y-periodic. The matrix sector, Y_3, is considered as the dual porosity, i.e. $K^\varepsilon_{ij}(y) \approx \varepsilon^2$, $y \in Y_3$.

The limit homogenized model derived from (17) involves macroscopic displacements, $\boldsymbol{u}$ and two pressures, p_1, p_2, which are associated with (macroscopic) *parallel flows* in the "arterial" and "venous" porosities. In analogy with the situation treated in Section 2., the macroscopic model involves homogenized parameters which depend on corrector basis functions – solutions of the local problems which are now defined.

The steady state correctors. Couples $(\bar{\boldsymbol{\omega}}^{rs}, \bar{\pi}^{rs})$ and $(\boldsymbol{\omega}^{*,\alpha}, \tilde{\pi}^\alpha(0))$ are solutions to the following problems (where $(\psi,\, \phi)_{Y_3}$ is the inner product of ψ and ϕ in $L^2(Y_3)$):

- Find $\bar{\boldsymbol{\omega}}^{rs} \in \mathbf{H}^1_\#(Y)$ and $\bar{\pi}^{rs} \in H^1_{\#0}(Y_3)$ so that

$$
\begin{aligned}
a_Y\left(\bar{\boldsymbol{\omega}}^{rs},\, \boldsymbol{v}\right) = -a_Y\left(\boldsymbol{\Pi}^{rs},\, \boldsymbol{v}\right) \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\#(Y) \;, \\
c_{Y_3}\left(\bar{\pi}^{rs},\, \psi\right) = -\left(\psi,\, \mathrm{div}_y \bar{\boldsymbol{\omega}}^{rs} + \mathrm{div}_y \boldsymbol{\Pi}^{rs}\right)_{Y_3} \quad \forall \psi \in H^1_{\#0}(Y_3) \;,
\end{aligned}
\tag{18}
$$

  where $\Pi^{rs}_i = \delta_{ri} y_s$, so that $\Pi^{rs}_i e^x_{rs}$ is the displacement induced in Y by locally uniform (macroscopic) strain e^x_{rs}.

- Find $\boldsymbol{\omega}^{*,\alpha} \in \mathbf{H}^1_\#(Y)$ and $\tilde{\pi}^\alpha_0(0) \in H^1_{\#0}(Y_3)$ such that

$$
\begin{aligned}
a_Y\left(\boldsymbol{\omega}^{*,\alpha},\, \boldsymbol{v}\right) - \left(\tilde{\pi}^\alpha_0(0),\, \mathrm{div}_y \boldsymbol{v}\right)_{Y_3} = -\!\!\!\!\!\!\int_{\Gamma_\alpha} \boldsymbol{v} \cdot \boldsymbol{n}^{[\alpha]} dS + \left(\tilde{\pi}^\alpha_1,\, \mathrm{div}_y \boldsymbol{v}\right)_{Y_3} \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\#(Y) \;, \\
\left(\psi,\, \mathrm{div}_y \boldsymbol{\omega}^{*,\alpha}\right)_{Y_3} = 0 \quad \forall \psi \in H^1_{\#0}(Y_3) \;,
\end{aligned}
\tag{19}
$$

where $\pi_1^\alpha \in H^1_\#(Y_3)$ is arbitrary satisfying $\pi_1^\alpha = \delta_{\alpha\beta}$ on Γ_β and $\boldsymbol{n}^{[\alpha]}$ is the unit normal outward to Y_α. It holds that $\boldsymbol{\omega}^{*,1} = -\boldsymbol{\omega}^{*,2}$ and $\tilde{\pi}^1(0) = 1 - \tilde{\pi}^2(0)$. Note $\tilde{\pi}^\alpha(0)$ serving as the Lagrange multiplier of the incompressibility of $\boldsymbol{\omega}^{*,\alpha}$ in Y_3.

The time-variant correctors satisfy the following general form of the evolutionary problem: Find $(\tilde{\boldsymbol{\omega}}, \tilde{\pi}) \in \mathbf{H}^1_\#(Y) \times H^1_{\#0}(Y_3)$ such that for $t > 0$

$$
\begin{aligned}
a_Y\left(\tilde{\boldsymbol{\omega}}(t),\, \boldsymbol{v}\right) - \left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}(t),\, \mathrm{div}_y \boldsymbol{v}\right)_{Y_3} &= 0 \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\#(Y)\;, \\
\left(\psi,\, \mathrm{div}_y \tilde{\boldsymbol{\omega}}(t)\right)_{Y_3} + c_{Y_3}\left(\tilde{\pi}(t),\, \psi\right) &= g(\psi) \quad \forall \psi \in H^1_{\#0}(Y_3)\;,
\end{aligned}
\tag{20}
$$

with the initial condition on $\tilde{\pi}(0)$. This generic problem is identified for two types of:

- ◦ strain-associated correctors $(\tilde{\boldsymbol{\omega}}^{rs}, \tilde{\pi}^{rs})$ solving (20) for $\tilde{\boldsymbol{\omega}}^{rs} \equiv \tilde{\boldsymbol{\omega}}$, $\tilde{\pi}^{rs} \equiv \tilde{\pi}$, $\tilde{\pi}^{rs}(0) := -\bar{\pi}^{rs}$, $g(\psi) \equiv 0$;
- ◦ pressure-associated correctors $(\tilde{\boldsymbol{\omega}}^\alpha, \tilde{\pi}_0^\alpha)$ solving (20) for $\tilde{\boldsymbol{\omega}}^\alpha \equiv \tilde{\boldsymbol{\omega}}$, $\tilde{\pi}_0^\alpha \equiv \tilde{\pi}$, $\tilde{\pi}_0^\alpha(0) := \tilde{\pi}^\alpha(0)$ (obtained by (19)), $g(\psi) \equiv -c_{Y_3}\left(\pi_1^\alpha,\, \psi\right)$. Then we define $\tilde{\pi}^\alpha(t) = \tilde{\pi}_0^\alpha(t) + \pi_1^\alpha$ for $t \geq 0$.

3.1. Homogenized problem

The macroscopic (homogenized) model involves the following homogenized material parameters which are evaluated using the characteristic responses obtained on solving (18)-(20):

- the homogenized *elastic* tensor

$$
\mathcal{E}_{ijkl} = a_Y\left(\boldsymbol{\Pi}^{kl} + \bar{\boldsymbol{\omega}}^{kl},\, \boldsymbol{\Pi}^{ij} + \bar{\boldsymbol{\omega}}^{ij}\right)\;,
$$

- the homogenized *viscosity* tensor of the fading memory

$$
\mathcal{H}_{ijkl}(t) = c_{Y_3}\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^{kl}(t),\, \bar{\pi}^{ij}\right)\;,
$$

- the homogenized *Biot-type* coefficients (related to local difference in pressures, $p_2 - p_1$)

$$
\begin{aligned}
\bar{\mathcal{P}}_{ij}^\alpha &= \left[\left(\tilde{\pi}^\alpha(0),\, \delta_{ij}\right)_{Y_3} - a_Y\left(\boldsymbol{\omega}^{*,\alpha},\, \Pi^{ij}\right)\right]\;, \\
\tilde{\mathcal{R}}_{ij}^\alpha(t) &= \left[\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\tilde{\pi}^\alpha(t),\, \delta_{ij}\right)_{Y_3} - a_Y\left(\tilde{\boldsymbol{\omega}}^\alpha(t),\, \boldsymbol{\Pi}^{ij}\right)\right]\;,
\end{aligned}
$$

- the homogenized *Barenblatt* coefficients, related to the flow between the two channels, induced by $p_\alpha - p_\beta$, $\beta \neq \alpha$. They describe the following two phenomena: *net fluid exchange through rigid-fixed interface* ($\tilde{w}_i^\alpha = -K_{ij}^3 \partial_j \tilde{\pi}^\alpha$ is the perfusion velocity)

$$
\tilde{\mathcal{G}}_+(t) = \fint_{\Gamma_1} \left(\tilde{\boldsymbol{\omega}}^1(t) + \tilde{\boldsymbol{w}}^1(t)\right) \cdot \boldsymbol{n}^{[1]} dS\;,
\tag{21}
$$

  and effects of the *incompressible interface sector*, the change of proportion between volumes of the channels Y_1 and Y_2 due to the compliant, incompressible interface Y_3

$$
\mathcal{G}^* = \fint_{\Gamma_\alpha} \boldsymbol{\omega}^{*,\alpha} \cdot \boldsymbol{n}^{[\alpha]} dS\;.
$$

- the homogenized *permeability* $\mathcal{C}^{\alpha}_{ij}$ of channel $\alpha = 1, 2$ is computed according to (8)-(9), where Y_c is replaced by Y_α.

The macromodel defined in terms of the homogenized coefficients involves macroscopic displacements, $\boldsymbol{u}(t) \in \boldsymbol{V} \subset \mathbf{H}^1(\Omega)$, and two macroscopic pressures, $p_1(t), p_2(t) \in H^1_0(\Omega)$; these satisfy the equilibrium equation

$$
\begin{aligned}
&\int_\Omega \left[\mathcal{E}_{ijkl} e^x_{kl}(\boldsymbol{u}(t)) + \int_0^t \mathcal{H}_{ijkl}(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\,\tau} e^x_{kl}(\boldsymbol{u}(\tau))\, d\tau \right] e^x_{ij}(\boldsymbol{v}) \\
&- \int_\Omega e^x_{ij}(\boldsymbol{v}) \int_0^t \tilde{\mathcal{R}}^1_{ij}(t-\tau)[p_1(\tau) - p_2(\tau)]\, d\tau \\
&- \sum_{\alpha=1,2} \int_\Omega \left[\frac{|Y_\alpha|}{|Y|} \delta_{ij} + \bar{\mathcal{P}}^{\alpha}_{ij} \right] p_\alpha(t)\, e^x_{ij}(\boldsymbol{v}) = L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}_0 \; ,
\end{aligned}
\tag{22}
$$

(where $\boldsymbol{V}_0$ is the space of the test displacements and $L(\cdot)$ is the load functional) and the two balance-of-mass equations for $\alpha, \beta = 1, 2,\ \beta \neq \alpha$

$$
\begin{aligned}
&\int_\Omega \mathcal{C}^{\alpha}_{ij}\, \partial^x_j p_\alpha(t)\, \partial^x_i q + \int_\Omega q\, \mathcal{G}^* \frac{\mathrm{d}}{\mathrm{d}\,t} \left(p_\alpha(t) - p_\beta(t) \right) \\
&+ \int_\Omega q \int_0^t \tilde{\mathcal{G}}_+(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\,\tau} \left(p_\alpha(\tau) - p_\beta(\tau) \right)\, d\tau \\
&+ \int_\Omega q \int_0^t \tilde{\mathcal{R}}^{\alpha}_{ij}(t-\tau) \frac{\mathrm{d}}{\mathrm{d}\,\tau} e^x_{ij}(\boldsymbol{u}(\tau))\, d\tau \\
&+ \int_\Omega q \left[\frac{|Y_\alpha|}{|Y|} \delta_{ij} + \bar{\mathcal{P}}^{\alpha}_{ij} \right] \frac{\mathrm{d}}{\mathrm{d}\,t} e^x_{ij}(\boldsymbol{u}(t)) = 0 \; , \quad \forall q \in H^1_0(\Omega) \; ,
\end{aligned}
\tag{23}
$$

which govern the fluid flows in the two channels and its redistribution between them.

4. Concluding remarks

The research reported here is related to our efforts in multiscale modeling of heterogeneous materials, namely with applications in tissue biomechanics. The homogenization technique allows for understanding how the effective material properties (coefficients) depend on interactions in the "microstructure" (i.e. the diffusion flow coupled with deformations in the present examples). Moreover, the two-scale modeling enables to compute quantities of interest at the "macroscopic" level and to employ for subsequent recovery of quantities characterizing tissue behaviour at the microscopic scale. In the context of the biomechanical applications, such option seems to be indispensable for modeling tissue growth and remodeling processes.

These models can be extended for treatment of large deformations, see e.g. Cimrman & Rohan (2007) for ad hoc parallel flows modeling, or Rohan (2006) for homogenization of the microflow modeling in soft tissues. Concerning application in *blood perfusion*, in real tissues the channel (blood vessel) geometries are characterized by *branching*, which is not completely consistent with assumptions of periodicity. Therefore, in Rohan (2008) homogenization of layered structures was suggested, where the periodicity assumption is relaxed and the model is well suited to capture the geometrical features the vascular "perfusion tree". The compact bone tissue model allows to obtain some microflow figures as the response on the macroscopic deformation. This will contribute to development of models which consider coupling of deformation induced flow in lacuno-canalicular porosities with electrochemical processes in living bone tissue, cf. Lemaire et al. (2008).

Both the homogenized models are implemented in our in-house developed SfePy code, Cimrman et al. (2008), which integrates all subroutines form solving the local corrector problems, evaluation of homogenized coefficients and solving the evolutionary macroscopic problems featured by occurrence of convolution integrals.

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