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MAGNETO-HYPERELASTIC MATERIAL IN A UNIFORM MAGNETIC FIELD: FEM CALCULATION OF STRESS AND STRAIN

S. T. Hoang, B. Marvalova*

Abstract: *Magneto-sensitive (MS) elastomers are a class of smart materials whose mechanical properties are changed instantly and reversibly by the application of an external magnetic field. These smart materials typically consist of micron-sized ferrous particles dispersed in a hyperelastic elastomer matrix (such as rubber, silicon rubber, etc.). The equations of the mechanical equilibrium and Maxwell's equations in an applied stationary magnetic field are summarized and constitutive equations describing the behavior of MS elastomer and a strain-energy function for isotropic MS elastomers are presented. FEM simulations are presented to illustrate the response of the magnetoelastic material to simultaneous application of the magnetic field and the mechanical loading.*

1. Introduction

In recent years, a number of industrial applications of MS elastomers have been developed. They provide relatively simple and quiet variable-stiffness devices as rapid-response interfaces between electronic and mechanical systems including controllable membranes, controllable stiffness devices and applications for the active control of structural components. MS elastomers have attracted considerable study of the interaction between magnetic and deformation fields in ferromagnetic materials and an increased need for reliable constitutive equations and appropriate strain-energy functions to model the magnetoelastic properties of these materials for use in the analysis and solution of representative boundary-value problems. We refer to the recent series of papers by Dorfmann, Ogden, Brigadnov and Bustamante (2003-2007) for the relevant theoretical background based on the general theory of nonlinear magnetoelasticity. The analytical solutions of boundary-value problems can be found in these papers, but they are mostly confined to infinite geometry and the solution do not reflect the magnetic field modification during the large deformation of material.

The purpose of the present paper is to implement the constitutive equations of MS elastomer into COMSOL MULTIPHYSICS finite element code with supporting Moving Mesh method which involves large geometric changes of the domains. The FE simulations of the magneto-mechanical coupling response of the magnetoelastic material are presented in which a homogeneous incompressible isotropic magnetoelastic material capable of large deformations is embedded in the uniform magnetic field and subjected to a simple shear stress.

* Ing. Sy Tuan Hoang, doc. Ing. Bohdana Marvalova, CSc.: Technical university of Liberec, Studentska 2, 46117 Liberec, Czech Republic, e-mail: Hoang.Sy.Tuan@tul.cz

The outline of this paper is as follows. In Section 2, we summarize the basic balance equations, constitutive equations and the free energy function which describe the behavior of magnetoelastic elastomers. Then in Section 3 we utilize the theory in COMSOL MULTIPHYSICS to investigate a particular magnetoelastic problem that is a pure plane shear deformation of the block embedded in a uniform magnetic field. Finally, in Section 4, we shall give some obtained conclusions.

2. Basic equations

In this section the equations for nonlinear magnetoelastic deformations, as developed by Bridgadnov and Dorfmann (2003), Dorfmann and Ogden (2005), are summarized.

We consider a magnetoelastic body in an undeformed configuration $\mathcal{B}_0$, with boundary $\partial\mathcal{B}_0$. A material point within the body in that configuration is identified by its position vector **X**. By the combined action of applied mechanical loads and magnetic fields, the material is then deformed from $\mathcal{B}_0$ to the configuration $\mathcal{B}$, with boundary $\partial\mathcal{B}$, so that the particle located at **X** in $\mathcal{B}_0$ now occupies the position $\mathbf{x} = \chi(\mathbf{X})$ in the deformed configuration $\mathcal{B}$. The function χ describes the static deformation of the body and is a one-to-one, orientation-preserving mapping with suitable regularity properties. The deformation gradient tensor **F** relative to $\mathcal{B}_0$ and its determinant are

$$\mathbf{F} = Grad\,\chi, \quad J = \det\mathbf{F} > 0 \tag{2.1}$$

2.1. Magnetic balance equations

In the Eulerian description, Maxwell's equations in the absence of time dependence, free charges and free currents reduce to

$$div\mathbf{B} = 0, \qquad curl\mathbf{H} = \mathbf{0} \tag{2.2}$$

which hold both inside and outside a magnetic material Kovetz (2000), where div and curl relates to $\mathcal{B}$. Thus, **B** and **H** can be regarded as fundamental field variables. In the vacuum, we have a basic relation between **B** and **H** as

$$\mathbf{B} = \mu_0\mathbf{H} \tag{2.3}$$

where $\mu_0 = 4\pi\times10^{-7}$ is a universal constant.

Associated with the equations (2.2) are the boundary continuity conditions

$$[\mathbf{B}].\mathbf{n} = 0, \qquad [\mathbf{H}]\times\mathbf{n} = \mathbf{0} \tag{2.4}$$

where [•] signifies a discontinuity across the boundary and **n** is its outward unit normal.

Lagrangian counterparts of **B** and **H**, denoted $\mathbf{B}_l$ and $\mathbf{H}_l$, respectively, are defined by

$$\mathbf{B}_l = J\mathbf{F}^{-1}\mathbf{B}, \qquad \mathbf{H}_l = \mathbf{F}^T\mathbf{H} \tag{2.5}$$

where the superscript ^{T} denotes the transpose of a tensor.

And these quantities equations (2.2) become

$$Div\mathbf{B}_l = 0, \qquad Curl\mathbf{H}_l = \mathbf{0} \tag{2.6}$$

where Div and Curl are the div and curl operators in $\mathcal{B}_0$, respectively.

The boundary conditions (2.4) can also be expressed in Lagrangian form

$$\left(\mathbf{B}_l - J\mathbf{F}^{-1}\mathbf{B}_0\right).\mathbf{n} = 0, \quad \left(\mathbf{H}_l - \mathbf{F}^T\mathbf{H}_0\right)\times\mathbf{n} = \mathbf{0} \tag{2.7}$$

in which $\mathbf{B}_0$ and $\mathbf{H}_0$ are the corresponding fields exterior to the material, but evaluated on the boundary $\partial\mathcal{B}_0$.

2.2. Mechanical balance equations

Let ρ_0 and ρ be the mass densities of the material in the reference and spatial configurations, $\mathcal{B}_0$ and $\mathcal{B}$, respectively. Then, the conservation of mass equation can be written simply as

$$\rho_0 = J\rho \tag{2.8}$$

The influence of the magnetic field on the mechanical stress in the deforming body may be incorporated through magnetic body forces or through a magnetic stress tensor (see Brigadnov and Dorfmann (2003)). Here, we use the latter approach and denote the resulting total Cauchy stress tensor by $\boldsymbol{\tau}$, which has the advantage of being symmetric. In case the absence of mechanical body forces, the equilibrium equation for a magnetoelastic solid in Eulerian configuration has the form

$$div\boldsymbol{\tau} = \mathbf{0} \tag{2.9}$$

By using the total nominal stress tensor, here denoted **T,** which is related to τ by

$$\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\tau} \tag{2.10}$$

then the equilibrium equation (2.9) may be expressed in Lagrangian form as

$$Div\mathbf{T} = \mathbf{0} \tag{2.11}$$

The boundary condition involving the stress τ, where traction rather than displacement is specified, may be written in the form

$$[\boldsymbol{\tau}]\mathbf{n} = \mathbf{0} \tag{2.12}$$

and we note that the traction $\boldsymbol{\tau}\mathbf{n}$ on the outer boundary includes a contribution from the (symmetric) Maxwell stress outside the material as well as any mechanical traction applied to the surface of the body.

We recall Brigadnov and Dorfmann (2003) that the Maxwell stress outside the material, denoted τ_M, is given by

$$\boldsymbol{\tau}_M = \mathbf{H}^* \otimes \mathbf{B}^* - \frac{1}{2}\left(\mathbf{H}^*.\mathbf{B}^*\right)\mathbf{I} \tag{2.13}$$

where $\mathbf{I}$ is the identity tensor and $\mathbf{B}^*$ and $\mathbf{H}^*$ are the corresponding fields exterior to the material evaluated on the boundary $\partial\mathcal{B}$, $\mathbf{B}^* = \mu_0\mathbf{H}^*$.

2.3. Constitutive equations

In this paper we consider formulations of the constitutive law based on $\mathbf{B}_l$ as the independent magnetic variable and the 'total' free energy function is expressed as

$$\Omega = \Omega\left(\mathbf{F}, \mathbf{B}_l\right) \tag{2.14}$$

in which Ω is treated as a function of **F** and $\mathbf{B}_l$. From Clausius–Duhem inequality for electromagnetic media Brigadnov and Dorfmann (2003) we can derive constitutive equations which are related to the deformation gradient **F** and the magnetic induction $\mathbf{B}_l$. Then the total nominal stress **T** and the magnetic field $\mathbf{H}_l$ are given by the simple formulas

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}}, \qquad \mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l} \tag{2.15}$$

and for an incompressible material by

$$\mathbf{T} = \frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{F}^{-1}, \qquad \mathbf{H}_l = \frac{\partial \Omega}{\partial \mathbf{B}_l} \tag{2.16}$$

where p is a Lagrange multiplier associated with the constraint det**F** = 1.

The corresponding Eulerian quantities are given by

$$\boldsymbol{\tau} = J^{-1}\mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}}, \qquad \mathbf{H} = \mathbf{F}^{-T}\frac{\partial \Omega}{\partial \mathbf{B}_l} \tag{2.17}$$

and

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega}{\partial \mathbf{F}} - p\mathbf{I}, \ \mathbf{H} = \mathbf{F}^{-T}\frac{\partial \Omega}{\partial \mathbf{B}_l} \tag{2.18}$$

where **I** is the identity tensor.

The magnetic field direction is applied in the material by analogy to the preferred direction in a transversely isotropic elastic solid. Thus for isotropic materials, the energy function depends only on **C**, where $\mathbf{C}=\mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green deformation tensor, and $\mathbf{B}_l \otimes \mathbf{B}_l$, Dorfmann (2005) through the six invariants

$$\begin{aligned} &I_1 = tr\mathbf{C}, && I_2 = \frac{1}{2}\left[\left(tr\mathbf{C}\right)^2 - tr\left(\mathbf{C}^2\right)\right] \\ &I_3 = \det \mathbf{C} = J^2, && I_4 = \mathbf{B}_l.\mathbf{B}_l \\ &I_5 = \mathbf{B}_l.\left(\mathbf{C}\mathbf{B}_l\right), && I_6 = \mathbf{B}_l.\left(\mathbf{C}^2\mathbf{B}_l\right) \end{aligned} \tag{2.19}$$

For incompressible materials, $I_3=1$ and only the five invariants I_1, I_2, I_4, I_5, and I_6 remain. The total stress tensor $\boldsymbol{\tau}$ is then expressed as

$$\tau = -p\mathbf{I} + 2\Omega_1\mathbf{b} + 2\Omega_2\left(I_1\mathbf{b} - \mathbf{b}^2\right) + 2\Omega_5\mathbf{B}\otimes\mathbf{B} + 2\Omega_6\left(\mathbf{B}\otimes\mathbf{b}\mathbf{B} + \mathbf{b}\mathbf{B}\otimes\mathbf{B}\right) \tag{2.20}$$

where $\Omega_i = \partial\Omega/\partial I_i$, and the total nominal stress tensor **T** as

$$\begin{aligned} \mathbf{T} = &-p\mathbf{F}^{-1} + 2\Omega_1\mathbf{F}^T + 2\Omega_2\left(I_1\mathbf{F}^T - \mathbf{F}^T\mathbf{b}\right) \\ &+2\Omega_5\mathbf{B}_l\otimes\mathbf{B} + 2\Omega_6\left(\mathbf{B}_l\otimes\mathbf{b}\mathbf{B} + \mathbf{F}^T\mathbf{B}\otimes\mathbf{B}\right) \end{aligned} \tag{2.21}$$

Finally, the magnetic field vector **H** is found from $(2.17)_2$ as

$$\mathbf{H} = 2\left(\Omega_4\mathbf{b}^{-1}\mathbf{B} + \Omega_5\mathbf{B} + \Omega_6\mathbf{b}\mathbf{B}\right) \tag{2.22}$$

and its Lagrangian counterpart is

$$\mathbf{H}_1 = 2\left(\Omega_4\mathbf{B}_1 + \Omega_5\mathbf{C}\mathbf{B}_1 + \Omega_6\mathbf{C}^2\mathbf{B}_1\right) \tag{2.23}$$

In order to simulate behaviors of the incompressible magnetoelastic elastomer, we refer and inherit a simple form of the free energy function as proposed in Dorfmann's paper (2005)

$$\Omega = \frac{G}{4}\left[c_1(I_1-3)+c_2(I_2-3)\right]+\frac{1}{\mu_0}(\alpha I_4+\beta I_5) \tag{2.24}$$

where $G = G_0\left(1+\eta|\mathbf{B}|^2\right)$ is the shear modulus in the reference configuration, G_0 is the field independent shear modulus (or zero-field modulus) and η, c_1, c_2, α and β are non-dimensional material constants to be determined experimentally. In the absence magnetic field, the expression (2.24) gives classical Mooney-Rivlin model.

3. Example of FEM simulation of the response of MS elastomer

3.1. Solution strategies

To implement the coupled interaction of magnetic field and mechanical material we assert the magnetic field is solved in the spatial coordinates and the deformation response is computed in the reference coordinates. We also suppose that the external boundaries of the surrounding space (vacuum) are far away from the surfaces of the MS elastomer body, hence the remote magnetic field is homogeneous.

The influence of the magnetic field on the surface of magnetoelastic body is expressed via the corresponding tractions in the material coordinates and defined as follow

$$\mathbf{T}_a = J\mathbf{F}^{-1}\boldsymbol{\tau}_a \tag{3.1}$$

where $\boldsymbol{\tau}_a$ is the corresponding traction evaluated in the spatial coordinate, and relates to the Maxwell stress $\boldsymbol{\tau}_M$ defined by eq. (2.13) by $\boldsymbol{\tau}_a = \boldsymbol{\tau}_M\mathbf{n}$, with $\mathbf{n}$ is a unit outer normal vector on the boundary of the deformed body.

The body is submitted to both magnetic and external forces, the stresses and strains of the body are calculated and the displacement components are simultaneously passed to the Moving Mesh interface so that the boundaries of magnetic field domain change in compliance with the large deformation of body. So we can easily investigate the effect of the magnetic field to the magnetoelastic material during the mechanical loading or vice versa.

3.2. FEM solutions for magnetoelastic material

In following simulations a plate of uniform thickness subjected to a unidirectional quasi-static shear deformation along the x-direction is considered. The maximum magnitude of the shear stress applied at the plate boundary comes up to $0.2G_0$, where G_0 is a shear modulus of the plate in the absence of magnetic field. The uniform magnetic induction field **B** acts across the plate and it changes from 0 to 1T.

We assume that the magnetosensitive elastomer is described by the strain-energy function (2.24). The magnetic field inside the material is defined from the strain-energy function by eq. (2.22) and interconnected to the outer magnetic field through the jump conditions (2.4). The total nominal stress tensor (2.21) in the material is calculated by differentiation of the strain-energy function with respect to **F** and satisfies the boundary conditions with the prescribed displacements or the tractions comprising both applied and magnetic forces. All the constitutive relations have been implemented into Comsol Multiphysics FE code.

The chosen form of the strain-energy function (2.23) and the material parameters Dorfmann (2003, 2005) and Bustamante (2007) are

$$\begin{array}{lll} \alpha = 0,05 \div 0,15 & \beta = 0,2 & G_0 = 0,25MPa \\ \eta = 0,9 & c_1 = 0,4 & c_2 = 1,6 \end{array} \qquad (3.2)$$

Firstly, we compare the dependency of displacement on the magnetic field in two cases - with and without use of the Moving Mesh. The results are depicted in Fig.1. The difference in the two cases is quite significant, so the influence of the change of magnetic domain boundaries is considerable and should not be neglected.

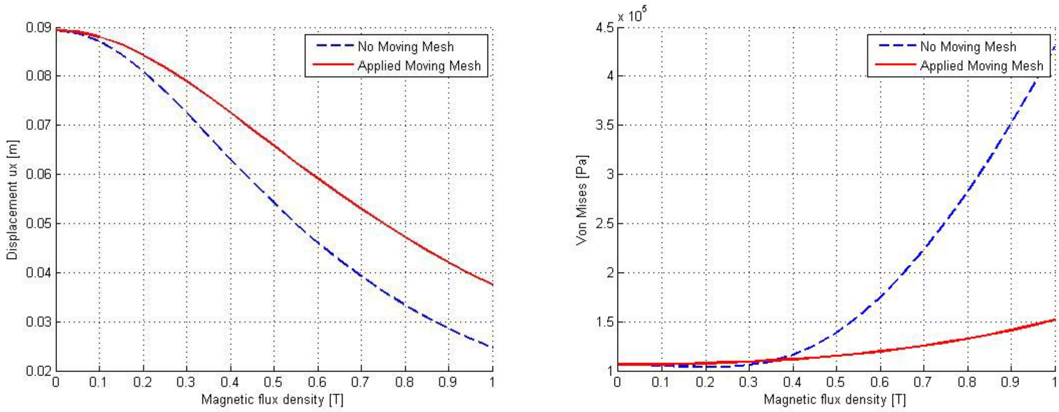

Fig.1 Displacement and stress in two cases: a) no Moving Mesh, b) applied Moving Mesh

In Fig.2 there is a distribution of magnetic field inside and outside the body while the body is subjected to an external shear stress at the top with the magnitude by $0.2G_0$ (0.05MPa) and embedded in a uniform and stationary magnetic field at applied magnetic flux $B_y = 0.3$T. Owing to the change of the domains induced by the deformation the distribution of magnetic field is not symmetric.

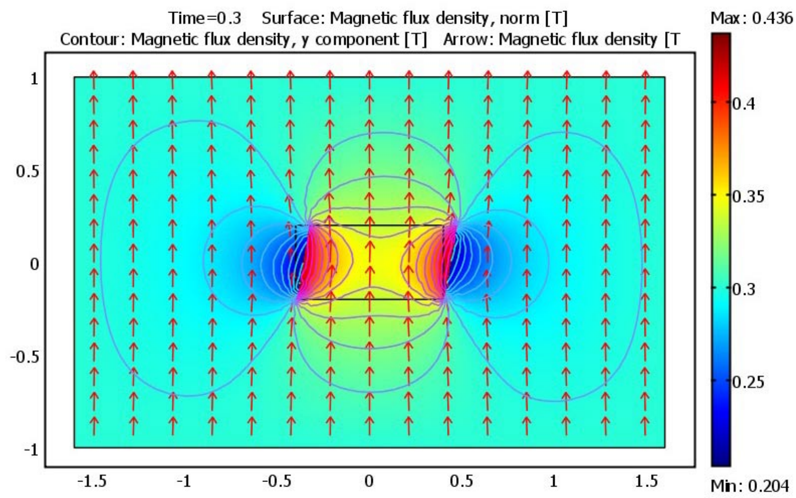

Fig.2 Distribution of magnetic field inside and outside the material

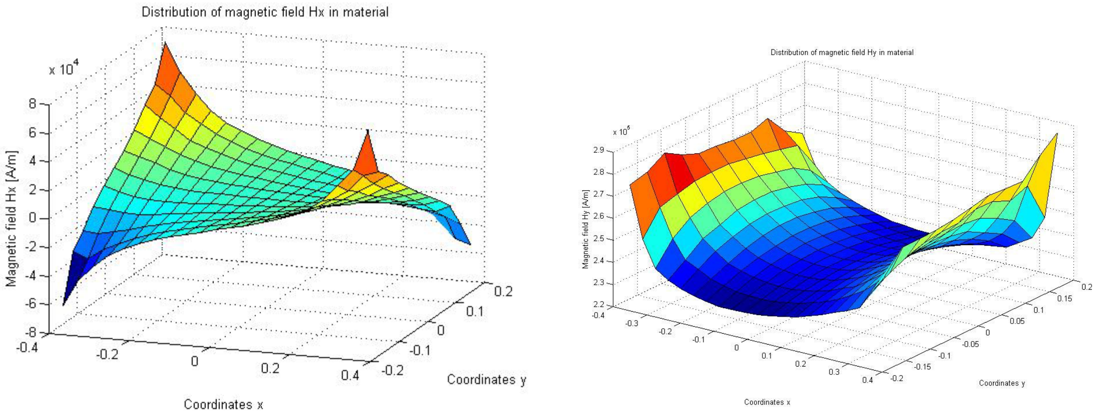

Fig.3 Variation of magnetic fields H_x, H_y in the deformation body

The variation of magnetic fields within the material are shown in Fig.3, it is similar to Bustamante (2007) results. The field component H_x perpendicular to the direction of the applied field is antisymmetric and the magnetic field strongly varies close to the body edges.

The magnetic tractions acting on the body and the stress are displayed at Fig.4. The direction of the magnetic tractions implies the body tends to lengthen along the direction of the applied field.

In Fig.5 the dependency of the stress components on the applied magnetic field is presented. The field affects strongly the increasing normal stresses s_x and s_y, while the component of shear stress is nearly unchanged.

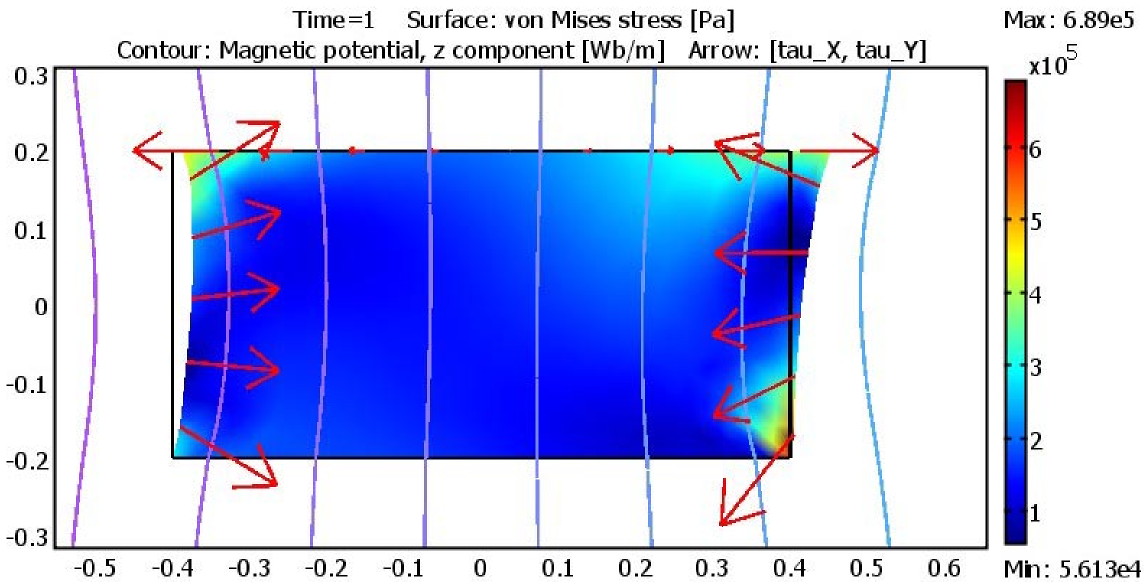

Fig.4 Magnetic traction and Von Mises stress

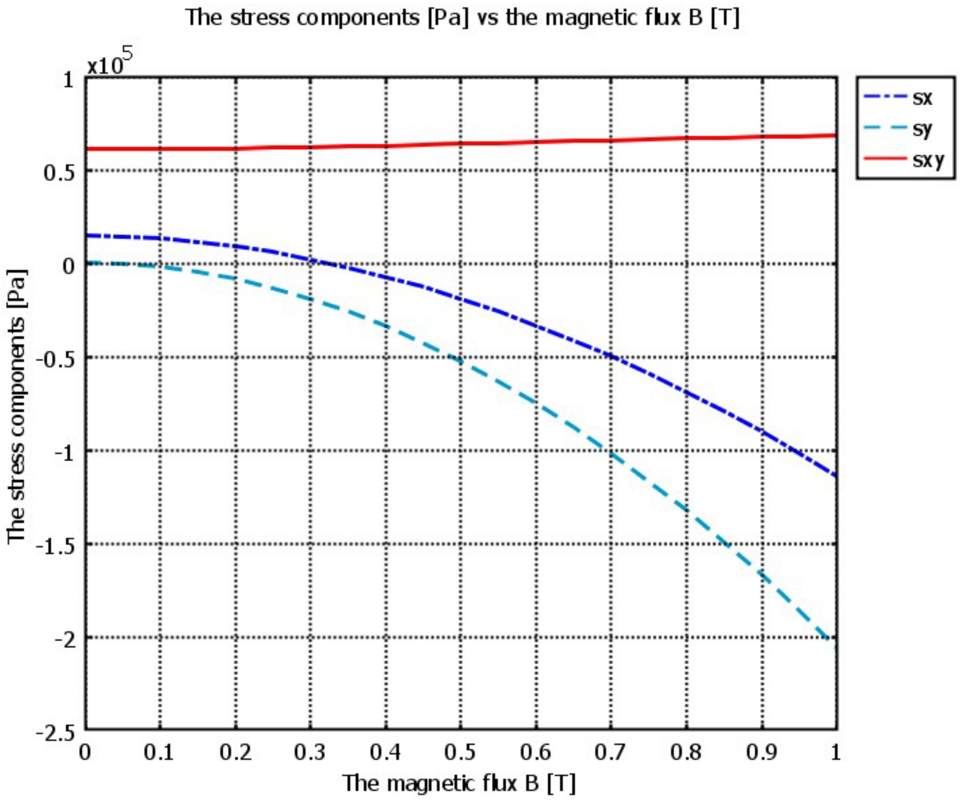

Fig.5 Dependency of stress components on magnetic field

Finally, in order to see a role of the constant α we investigate the deformation and the change of the stress with the increasing magnetic field at different values of α, in which α=0.05, 0.08, 0.1, 0.15, as in Fig.6. The effect of the coefficient α to the deformation is relatively small, but for the variation of the stress is quite large. In addition, the deformation of the body decreases with the increasing magnetic field, so the material becomes stiffer with the increasing magnetic field.

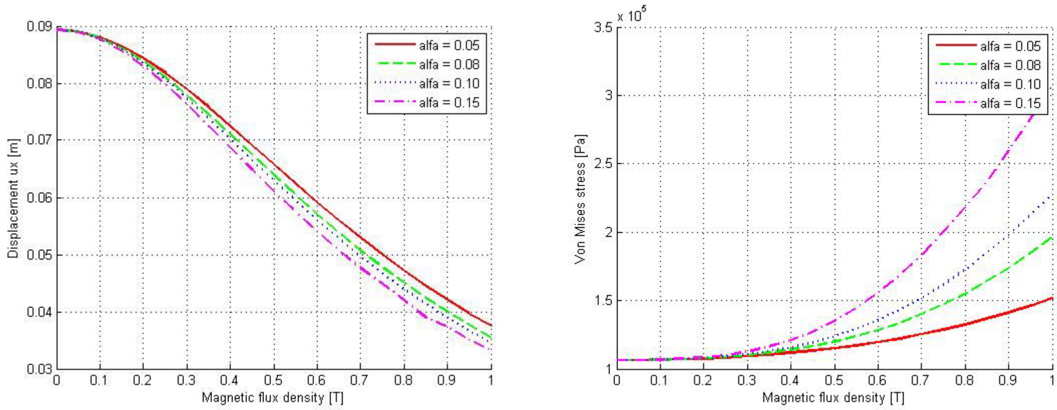

Fig.6 Dependency of the deformation and Von Mises stress on the magnetic field and the coefficient α

4. Conclusions

The change of magnetic domain boundaries due to body large deformations has been included in computations of the coupled magneto-mechanical response of MS elastomer. The behavior of magnetoelastic composite and the distribution of magnetic field in the material are simulated in finite element code. The computed increasing stiffness of MS elastomer with the increasing magnetic field agrees with practical experimental results and with the predictions in the mentioned papers. The determination of material parameters will be the subject of further experimental measurements.

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