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GEOMETRY OF FINITE DEFORMATIONS, LINEARIZATION, AND INCREMENTAL DEFORMATIONS UNDER INITIAL STRESS / STRAIN

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Summary: *Stepwise integration of a finitely deformed body is based on an incremental kinematics of a medium under initial stress / strain conditions. As usual in continuum mechanics, deformation and stress tensors at a point are considered to form vector (i.e. Euclidean) spaces. The derivatives therefore are the usual ones, and when dealing with time evolution we have to resort to objective time derivatives. There are two classical textbooks dealing with the incremental approach – the book of Green & Zerna and the book of Biot, both approaching the subject from different perspectives. The first as an incremental strain, the second one as an incremental stress. Employing an approach of differential geometry to the description of finite deformations, resulting in a geometrically consistent linearization, we can simply compare these two descriptions. The contribution reviews the theory of finite deformations both from the finite-dimensional Riemannian geometry viewpoint (standard knowledge) and the infinite-dimensional Riemannian geometry viewpoint (non-standard, disputable approach), considering the space of deformation tensors as a constantly non-negatively curved space.*

1. Introduction

The focus of this presentation is an application of differential geometry to the description of finite deformations in continuum mechanics. The Riemannian geometry naturally enters the theory via deformation tensor fields. In fact, if we denote by g a Riemannian metric on actual configuration $\mathcal{S} = \Phi(\mathcal{B}) \subset \mathcal{E}^3$, i.e. sufficiently smooth symmetric positive-definite covariant tensor field of second order, which assigns to each point $x \in \mathcal{S}$ a metric tensor that determines a scalar product of any two vectors emanating from this point – and so a geometry in its vicinity, then the *covariant* form $\mathbf{C}^\flat$ of the well-known second order right Cauchy-Green *mixed* deformation tensor $\mathbf{C} = \mathbf{F}^\mathrm{T}\mathbf{F}$ can be shown to be an image of g in $\mathcal{B}$. Thus the right Cauchy-Green deformation tensor is again the Riemannian metric – but now on referential configuration $\mathcal{B}$, and describes the geometry of the deformed body $\mathcal{S}$ from the point of view of an observer attached to the undeformed body $\mathcal{B}$. Similar interpretation also applies to the left Cauchy-Green $\mathbf{b} = \mathbf{F}\mathbf{F}^\mathrm{T}$, the Piola $\mathbf{B} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}$, and the Almansi $\mathbf{c} = \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}$ deformation tensor fields.

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Moreover, differential geometry enables us to employ its tools for analysis of deformation process. For example, the rate-of-deformation tensor $\mathbf{d}$ can be expressed both in terms of the Lie derivative $2\mathbf{d}^\flat = \mathcal{L}_{v_t}\mathbf{g}$, and in terms of the covariant derivative $2\mathbf{d} = \nabla v_t + (\nabla v_t)^{\mathrm{T}}$.

In addition, we can move one step further and think of the finite deformation process as a trajectory $\mathbf{C}^\flat\colon I \to \mathcal{M}$ in the space $\mathcal{M} = \mathrm{Met}(\mathcal{B})$ of all Riemannian metrics on reference configuration $\mathcal{B}$. The image of the rate-of-deformation tensor $\mathbf{d}^\flat$ in referential configuration can be shown to be the time derivative $\partial\mathbf{C}^\flat_\mathbf{t}$, i.e. a vector to the curve $\mathbf{C}^\flat_\mathbf{t}$. Then all vectors emanating from a point $\mathbf{C}^\flat$ form a vector space $T_{C^\flat}\mathcal{M}$, and we prove that the deformation process within small deformations may be represented by a trajectory in this vector space. Introducing the scalar product on $T_{C^\flat}\mathcal{M}$ via $\mathbf{d}^\flat$ – $\partial\mathbf{C}^\flat_\mathbf{t}$ correspondence further edows $\mathcal{M}$ with the Riemannian structure, making it possible to introduce a covariant derivative, find a geometrical interpretation of the logarithmic strain and generalized it for strained initial configurations, as shown in Fiala (2007) based on Rougée (1997).

The central problem in finite deformations is the stress rate and the corresponding stress update algorithms in solving the finite deformation problems by incremental method. Within framework of $\mathcal{M}$ the stress field can be shown to be a covector (covariant vector), and evolving stress field during deformation process a covector field along the trajectory $\mathbf{C}^\flat_\mathbf{t}$. The covariant derivative is then the only tool capable of introducing the time derivative, in this case via geometrically consistent linearization. Incidentally, it proves to be the Zaremba-Jaumann objective time derivative just like in Biot (1965).

The space $\mathcal{M}$, as the space of metric tensors, corresponds to the space of symmetric positive definite matrices. Regarding this space as the Riemannian manifold (in fact constantly non-negatively curved space (see Bhatia (2007)), rather then a vector space, enables us to find geometrical interpretation of the logarithmic strain as a vector, and generalized it for strained initial configurations. Moreover, straight lines here prove to be matrix exponentials, and so instead of adding an increment to an initial deformation, it should be actually mapped to $\mathcal{M}$ by means of the matrix exponential, starting from an initial deformation. In the end, these conclusions will be compared with the approach of Green & Zerna and Biot.

2. Geometry of finite deformations

Our exposition of application of differential geometry to the theory of finite deformations draws on the book of Marsden & Hughes (1993) and papers of Sansour (1992), Svendsen & Tsakmakis (1994), Svendsen (1995), Giessen & Kollmann (1996), Stumpf & Hoppe (1997) and Kadianakis (1999). Now, let a body $\mathcal{B}$ occupy a connected region of the three-dimensional Euclidean space $\mathbb{E}^3$, from now on considered as the Riemannian manifold $\mathcal{E}^3$ (see Appendix for more information and further references). The region is assumed open and bounded, with smooth boundary. Simultaneously, the body $\mathcal{B}$ will represent a reference configuration, made up of points labelled by capital letters X. Points in ambient space – the Riemannian manifold $\mathcal{E}^3$, will be labelled by small letters x. Symbol $\mathcal{S} = \Phi(\mathcal{B})$ will denote an actual configuration.

First we will hightlight a fundamental difference between small – in fact infinitesimal, and finite deformations. The theory of small deformations approximates a deformation of a body in terms of infinitesimal displacement fields over an initial configuration of the body, whereas the theory of finite deformations describes it exactly in terms of differentiable invertible transformations – diffeomorphisms that transform the initial configuration into another, actual one. To be more illustrative, let us consider two successive deformations $X \to x_1 \to x_2$ and split them into the identity mapping plus a displacement field $x \equiv \Phi(X) = X + u(X)$ to obtain $x_2 = \Phi_2 \circ \Phi_1(X) = \Phi_2(x_1) = \Phi_2(X + u_1(X)) = X + u_1(X) + u_2(X + u_1(X))$. In the case of small deformations one neglects all the terms of second order in magnitude, and so the relation takes form $x_2 \approx X + u_1(X) + u_2(X)$, i.e. the diffeomorphism Φ is replaced by the displacement field u, as a correction to the identity mapping $x = \Phi_{Id}(X) \approx X$, and the concept of diffeomorphisms transforms into that of fields.

2.1. Deformation tensor fields

A ***deformation*** of a body, globally described in terms of an injective differentaible mapping (diffeomorphism) $\Phi : \mathcal{B} \to \mathcal{E}^3$, can be locally characterized by a deformation tensor fields – most frequently by the field of the *right Cauchy-Green deformation tensors* $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, but also by deformation fields of *left Cauchy-Green* $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, *Piola* $\mathbf{B} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}}$, or *Almansi* $\mathbf{c} = \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}$ tensors. We now show that these deformation fields are actualy Riemannian metrics.

Let us denote by g a Riemannian metric on $\mathcal{E}^3$, i.e. sufficiently smooth symmetric positive-definite covariant tensor field of second order, whose value at each point $x \in \mathcal{E}^3$ is a tensor that determines a scalar product of any two vectors emanating from this point, and so establishes a geometry in its vicinity. By G we fix a metric on reference configuration. If we apply a specification for the transposed deformation gradient $\mathbf{F}^{\mathrm{T}} = \mathbf{G}^{-1}\mathbf{F}^{*}\mathbf{g}$ by (87), then for the mixed RIGHT CAUCHY-GREEN *deformation tensor* field we obtain

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{G}^{-1}\mathbf{F}^{*}\mathbf{g}\,\mathbf{F} = \mathbf{G}^{-1}\,\boldsymbol{\Phi}^{*}(\mathbf{g}), \tag{1}$$

where $\boldsymbol{\Phi}^{*}(\mathbf{g})$ denotes a transformation (***pull-back***) of metric g from actual to referential configuration (see (92), now $\mathbf{F} = T\Phi$). Since $\mathbf{C}^{\flat} = \mathbf{G}\mathbf{C}$ by (84), we can transcript (1) into more simple, *covariant form* $\mathbf{C}^{\flat} = \boldsymbol{\Phi}^{*}(\mathbf{g})$, i.e. $C^{\flat} = \Phi^{*}(g)$ if we leave specific representation of Remark-2 (2-tensors are in general labeled in italic, their specific representation as linear mappings in bold). *The covariant form of the right Cauchy-Green deformation tensor field* $C^{\flat} = \Phi^{*}(g)$ *is thus a Riemannian metric on* $\mathcal{B}$, and because of

$$C^{\flat}(U,V) \equiv \langle \mathbf{C}^{\flat}U, V\rangle_{T_X\mathcal{B}} \equiv G(\mathbf{C}U,V) = g(u,v) \equiv \langle \mathbf{g}u, v\rangle_{T_x\mathcal{S}}\,, \tag{2}$$

it describes the geometry of the deformed body from the point of view of an observer attached to the undeformed body. For more about the scalar product of vectors (2), see relations (79) and (80) from Appendix. Here, vectors before deformation $U, V \in T_X\mathcal{B}$ turning into $u, v \in T_x\mathcal{S}$ after deformation are interrelated by $u = \Phi_{*}(U) \equiv \mathbf{F}U \circ \Phi^{-1}$ and $v = \Phi_{*}(V) \equiv \mathbf{F}V \circ \Phi^{-1}$ (89).

Similarly, for the field of mixed PIOLA *deformation tensor* field we have

$$\mathbf{B} = \mathbf{F}^{-1}\mathbf{F}^{-\mathrm{T}} = \mathbf{F}^{-1}\mathbf{g}^{-1}\mathbf{F}^{-*}\mathbf{G} = \boldsymbol{\Phi}^{*}(\mathbf{g}^{-1})\mathbf{G}\,, \tag{3}$$

where again $\boldsymbol{\Phi}^{*}(\mathbf{g}^{-1})$ stands for the pull-back of metric $\mathbf{g}^{\sharp}$ from actual to referential configuration – now metric in dual space of covectors, (see (92); moreover, in this representation $\mathbf{g}^{\sharp} \equiv \mathbf{g}^{-1}$). Since $\mathbf{B}^{\sharp} = \mathbf{B}\mathbf{G}^{-1}$ by (84), relation (3) can be put into *contravariant form*

$\mathbf{B}^\sharp = \mathbf{\Phi}^*(\mathbf{g}^{-1})$, i.e. $B^\sharp = \Phi^*(g^\sharp)$. Again we can express the scalar product of two covectors (see 81) in referential configuration in terms of Piola tensor – here covectors before deformation $A, D \in T_X^*\mathcal{B}$ turning into $a, d \in T_x^*\mathcal{S}$ after deformation, and so interrelated by $a = \Phi_*(A) = \mathbf{F}^{-*}A \circ \Phi^{-1}$ and $d = \Phi_*(D) = \mathbf{F}^{-*}D \circ \Phi^{-1}$ (see (90)), to obtain

$$B^\sharp(A, D) \equiv \langle A, \mathbf{B}^\sharp D\rangle_{T_X\mathcal{B}} \equiv G^\sharp(\mathbf{B}^* A, D) = g^\sharp(a, d) \equiv \langle a, \mathbf{g}^\sharp d\rangle_{T_x\mathcal{S}}\,, \tag{4}$$

where $\mathbf{B}^* = \mathbf{G}\mathbf{B}\mathbf{G}^{-1}$. *The Piola deformation field $B^\sharp = \Phi^*(g^\sharp)$ thus describes the geometry of the deformed body from the viewpoint of an observer attached to the reference configuration, now in terms of covectors.*

We can also proceed reversely and be interested in the geometry of the body $\mathcal{B}$ from the viewpoint of an observer in actual configuration $\mathcal{S}$. Now we obtain, making use of ***push-forward*** transformation $\mathbf{\Phi}_*$ (see (93)), the LEFT CAUCHY-GREEN *deformation tensor* field $b^\sharp = \Phi_*(G^\sharp)$

$$\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{F}\mathbf{G}^{-1}\mathbf{F}^*\mathbf{g} = \mathbf{\Phi}_*(\mathbf{G}^\sharp)\mathbf{g}\,, \tag{5}$$

or the ALMANSI *deformation tensor* field $c^\flat = \Phi_*(G)$

$$\mathbf{c} = \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1} = \mathbf{g}^{-1}\mathbf{F}^{-*}\mathbf{G}\mathbf{F}^{-1} = \mathbf{g}^{-1}\mathbf{\Phi}_*(\mathbf{G})\,. \tag{6}$$

Expressions for corresponding scalar products in reference configuration, being expressed in actual configuration, are analogous to previous ones (3) and (6). Namely,

$$b^\sharp(a, d) \equiv \langle a, \mathbf{b}^\sharp d\rangle_{T_x\mathcal{S}} \equiv g^\sharp(\mathbf{b}^* a, d) = G^\sharp(A, D) \equiv \langle A, \mathbf{G}^\sharp D\rangle_{T_X\mathcal{B}} \text{ and} \tag{7}$$

$$c^\flat(u, v) \equiv \langle \mathbf{c}^\flat u, v\rangle_{T_x\mathcal{S}} \equiv g(\mathbf{c}u, v) = G(U, V) \equiv \langle \mathbf{G}U, V\rangle_{T_X\mathcal{B}}\,. \tag{8}$$

From (7) and (8) we conclude that both $b^\sharp$ and $c^\flat$ describe geometry of the undeformed body from the viewpoint of an observer attached to the actual configuration. Again $\mathbf{b}^* = \mathbf{g}\,\mathbf{b}\,\mathbf{g}^{-1}$.

2.2. Strains and Logarithmic strains

Since both $C^\flat$ and G belong to the same tensor space, we can subtract G from $C^\flat$ to find a relative deformation (i.e. strain) – the GREEN-ST.VENANT *strain tensor* $E^\flat = \frac{1}{2}(C^\flat - G)$, resp. $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$. Similarly, we arrive at another Lagrangian strain tensor – the PIOLA *strain tensor* $H^\sharp = \frac{1}{2}(B^\sharp - G^\sharp)$, resp. $\mathbf{H} = \frac{1}{2}(\mathbf{B} - \mathbf{I})$. As for Eulerian strain tensors, we have the FINGER *strain tensor* $h^\sharp = \frac{1}{2}(g^\sharp - b^\sharp)$, resp. $\mathbf{h} = \frac{1}{2}(\mathbf{i} - \mathbf{b})$, and the ALMANSI-HAMMEL *strain tensor* $e^\flat = \frac{1}{2}(g - c^\flat)$, resp. $\mathbf{e} = \frac{1}{2}(\mathbf{i} - \mathbf{c})$.

There exist another Lagrangian and Eulerian strain tensors – the ***logarithmic strain tensors*** introduced by Hill (see for example Xiao et al. (1997)). As we shall see later, they constitute more convenient strain measures, since they properly reflect the specific geometry of the underlying space. If the deformation gradient $\mathbf{F}$ is expressed through polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, then $\mathbf{C} = \mathbf{U}^2$, $\mathbf{B} = \mathbf{U}^{-2}$, $\mathbf{b} = \mathbf{V}^2$, $\mathbf{c} = \mathbf{V}^{-2}$, and the logarithmic strains are defined as follow

$$\mathbf{L} = \log\mathbf{U} = \tfrac{1}{2}\log\mathbf{C} \tag{9}$$

$$\mathbf{l} = \log\mathbf{V} = \tfrac{1}{2}\log\mathbf{b}\,, \tag{10}$$

where for symmetric matrix $\mathbf{X} = \sum_i^3 \lambda_i \, N_i \otimes N_i$ its matrix logarithm is defined as

$$\log \mathbf{X} = \lim_{n \to 0} \frac{1}{n} \left(\mathbf{X}^n - \mathbf{I} \right) = \sum_i^3 \log(\lambda_i) \, N_i \otimes N_i \, . \tag{11}$$

Since $\mathbf{U}$ is symmetric positive-definite mixed 2-tensor field over $\mathcal{B}$, and $\mathbf{V}$ symmetric positive-definite mixed 2-tensor field over $\mathcal{S}$ ($\mathbf{F}$ and $\mathbf{R}$ are two-point tensor fields) it holds both

$$\mathbf{U}^n = \Phi^*(\mathbf{V}^n) \equiv \mathbf{F}^{-1} \, \mathbf{V}^n \mathbf{F} \; (= \mathbf{R}^{-1} \, \mathbf{V}^n \mathbf{R}) \tag{12}$$

$$\mathbf{V}^n = \Phi_*(\mathbf{U}^n) \equiv \mathbf{F} \mathbf{U}^n \mathbf{F}^{-1} \; (= \mathbf{R} \mathbf{U}^n \mathbf{R}^{-1}) \, , \tag{13}$$

and

$$\mathbf{L} = \Phi^*(\mathbf{l}) \equiv \mathbf{F}^{-1} \mathbf{l} \mathbf{F} \tag{14}$$

$$\mathbf{l} = \Phi_*(\mathbf{L}) \equiv \mathbf{F} \mathbf{L} \mathbf{F}^{-1} . \tag{15}$$

Later we will find geometric interpretation and corresponding generalization of these strain tensors.

2.3. Time-dependent deformation

Let us now consider a ***deformation process***, i.e. a time-parameterized smooth sequence of diffeomorphisms $\Phi_t : \mathcal{B} \to \mathcal{E}^3$, for time $t \geq t_0$. If we denote $\Psi_{t,\,s} \equiv \Phi_t \circ \Phi_s^{-1} : \mathcal{E}^3 \to \mathcal{E}^3$, then

$$\Psi_{t,s} = \Psi_{t,r} \circ \Psi_{r,s} \tag{16}$$

$$\Psi_{r,r} = \text{identity}$$

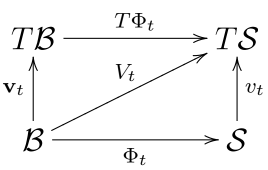
and

$$\frac{d}{dt} \, \Psi_{t,s} = v_t \circ \Psi_{t,s} \, , \tag{17}$$

where $v_t(x) = V_t \circ \Phi_t^{-1}(x)$ represents the Euler velocity defined by means of the Lagrange velocity V_t

$$V_t(X) := \left. \frac{\partial \Phi_t(X)}{\partial t} \right|_X . \tag{18}$$

Collection $\Psi_{t,s}$ is called the *flow* of v_t (see Marsden & Hughes (1993), p 95), where the Euler velocity v_t is a vector field over the actual configuration $\mathcal{S}$, and its corresponding vector field over the body $\mathcal{B}$ is called the convective velocity field $\mathbf{v}_t = \Phi_t^*(v_t)$. Diagram below highlights mutual relations among these three vector fields.



Linearization of the expression (17) results in time rate of the tangent mapping $T\Psi_{t,s}$ (see Marsden & Hughes (1993), p 95)

$$\frac{d}{dt} \, T\Psi_{t,s} = \widehat{\nabla} v_t \circ T\Psi_{t,s} \qquad (\text{or} \;\; \dot{\mathbf{F}} = \mathbf{L}\mathbf{F}) \, , \tag{19}$$

where the velocity gradient $\widehat{\nabla} v_t$ represents a linearized Euler vector field v_t. Expressed in components by means of covariant derivative $\widehat{\nabla}$ associated with metric g on $\mathcal{E}^3$, it reads

$$\widehat{\nabla} v_t = (v_t)^i|_j \; \partial x^i \otimes dx^j . \tag{20}$$

From (19) for a general time-dependent tensor field θ_t over $\mathcal{E}^3$ one obtains (see Marsden & Hughes (1993), p 95)

$$\left(\frac{d}{dt} \, \Psi^*_{t,s} \right) (\theta_t) = \Psi^*_{t,s} \left(\frac{\partial}{\partial t} \, \theta_t + \mathcal{L}_{v_t} \theta_t \right) , \tag{21}$$

and so

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{v_t} \right) (\theta_t) = \left(\Psi_{t,s\,*} \circ \frac{d}{dt} \circ \Psi^*_{t,s} \right) (\theta_t) \, , \tag{22}$$

where symbol $\mathcal{L}_{v_t}$ stands for the LIE DERIVATIVE with respect to a vector field v_t. Since we consider time-dependent tensor fields θ_t, the Lie derivative is modified by partial time derivative ∂_t to obtain (see Frankel (1997), Abraham et al. (1988) and Schutz (1999))

$$L_{v_t} \! := \frac{\partial}{\partial t} + \mathcal{L}_{v_t} \, . \tag{23}$$

This ***time derivative based on Lie derivative*** *expresses the rate of change of time-dependent tensor fields at individual points of a body, as seen by an observer in actual configuration, as a consequence of moving coordinate system.* The time derivative itself is a result of comparison of time dependent values of the tensor field always in one tensor space – the tensor space corresponding to a particular point of the body $X \in \mathcal{B}$, which is a simple task due to linear nature of the tensor space.

If on the other hand we try to calculate the Euler acceleration a_t still staying in the ambient space $\mathcal{E}^3$ we have to compare values of the vector field v_t, not only in different moments of time but also in different tangent spaces attached to different points of $\mathcal{E}^3$ through which the point X of the body passes. In order to do this, we have to resort to covariant derivative. In fact, the Euler acceleration $a_t(x) = A_t \circ \Phi_t^{-1}(x)$ is a vector field over actual configuration $\mathcal{S}$, again defined in terms of the Lagrange acceleration

$$A_t(X) \! := \left. \frac{\partial V_t(X)}{\partial t} \right|_X , \tag{24}$$

and its corresponding field over the body $\mathcal{B}$ forms the convective acceleration $\mathbf{a}_t = \Phi_t^*(a_t)$. For all these three vector fields a similar diagram as for velocities applies. If we express the Euler acceleration directly in terms of available tools of the space $\mathcal{E}^3$, we obtain relation (see Marsden & Hughes (1993))

$$a_t = \left(\frac{\partial}{\partial t} + \widehat{\nabla}_{\dot{c}} \right) v_t \, , \tag{25}$$

where $c(t) = \Phi_t(X)$ represents the trajectory of a given point X in space $\mathcal{E}^3$, $\dot{c} = v_t$ is its velocity, and $\widehat{\nabla}_{\dot{c}}\, \theta$ stands for the covariant derivative of any tensor field θ in $\mathcal{E}^3$ along the trajectory $c(t)$.

By definition, the *covariant derivative* expresses the rate of change of tensor quantity $\theta(x)$ at a point $x = c(t)$, when passing through it in a direction and velocity given by its tangent vector $v_t(x) = \dot{c}(t)$. Geometrically, the covariant derivative is closely related to the parallel translation of tensors along curves. During calculation one compares values of a tensor field from different tensor spaces corresponding to different points of the space $\mathcal{E}^3$. Namely, a tensor $\theta_{c(t)}$ at a point $c(t)$ have to be parallelly translated $\theta^{\leftarrow}_{c(t)}$ to the initial tensor space at the point $c(0)$ along the curve $c(t)$, in limit characterized by vector $\dot{c}$, where we can subtract them to compute the corresponding covariant derivative (for example Dodson & Poston (1997))

$$\widehat{\nabla}_{\dot{c}}\,\theta = \lim_{t\to 0} \frac{\theta^{\leftarrow}_{c(t)} - \theta_{c(0)}}{t}\,. \tag{26}$$

Since $\mathcal{E}^3$ is Euclidean space, parallel translation is independ of the selected path. If the manifold is curved, as is the case in our space $\mathcal{M}$, selection of curve matters. In the case of time-dependent tensor fields, the covariant derivative should be again modified by partial time derivative as in (23).

In this way we obtain one more time derivative – the ***time derivative based on covariant derivative****, which establishes the rate of change of time-dependent tensor fields θ_t along trajectory $c(t)$ when going through its points in the direction and velocity given by its tangent vector* $v_t = \dot{c}(t)$. Contrary to the time derivative based on Lie derivative, which is related to time changes of tensor fields in the body itself, even though interpreted from the viewpoint of actual configuration, the time derivative based on covariant derivative is related not only to time, but also to space variations of tensor fields when moving through the space $\mathcal{E}^3$.

For function f, its time derivative based on Lie derivative is

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{v_t}\right) f = \frac{\partial f}{\partial t} + v_t^i\,\frac{\partial f}{\partial x^i}\,, \tag{27}$$

i.e. it is the *material time derivative*, just like the time derivative based on covariant derivative for vector field u in Cartesian coordinates

$$\left[\left(\frac{\partial}{\partial t} + \widehat{\nabla}_{v_t}\right) u\right]^j = \frac{\partial u^j}{\partial t} + v_t^i\,\frac{\partial u^j}{\partial x^i}\,. \tag{28}$$

For other types of tensors, these derivatives look different.

2.4. Rate of deformation tensors

While the time derivative based on covariant derivative of the metric tensor g, resp $g^\sharp$ equals to zero (g is not only stationary, but it is a covariant constant $\widehat{\nabla} g = 0$), and so

$$\left(\frac{\partial}{\partial t} + \widehat{\nabla}_{v_t}\right) g = 0\,, \tag{29}$$

from the viewpoint of mechanics of continua its time derivative based on Lie derivative has clear mechanical meaning. In fact, it holds (see Frankel (1997), Marsden & Hughes (1993))

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{v_t}\right) g = \mathcal{L}_{v_t}\, g = 2\left[(\widehat{\nabla} v_t)^\flat\right]_{sym} = 2d^\flat(v_t)\,, \tag{30}$$

where

$$(\widehat{\nabla} v_t)^\flat = (v_t)_i|_j \, dx^i \otimes dx^j \tag{31}$$

and at the same time

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{v_t}\right) g^\sharp = \mathcal{L}_{v_t} \, g^\sharp = 2 \left[(\widehat{\nabla} v_t)^\sharp\right]_{sym} = -2 d^\sharp(v_t)\,, \tag{32}$$

where now

$$(\widehat{\nabla} v_t)^\sharp = (v_t)^i|^j \, \partial x^i \otimes \partial x^j. \tag{33}$$

Since

$$\partial C^\flat := \frac{\partial}{\partial t} \, C^\flat = 2 \Phi_t^*(d^\flat) \;\text{ and} \tag{34}$$

$$\partial B^\sharp := \frac{\partial}{\partial t} \, B^\sharp = -2 \Phi_t^*(d^\sharp), \tag{35}$$

an observer attached to the body $\mathcal{B}$ then interprets the ***rate-of-deformation tensor field*** $2d^\flat$*, resp.* $-2d^\sharp$*, as the* ***deformation rate*** $\partial C^\flat$*, resp.* $\partial B^\sharp$, and (cf. (2), (4))

$$\frac{d}{dt} g(u,v) = 2 d^\flat(u,v) = \partial C^\flat(U,V) \;\text{ and} \tag{36}$$

$$\frac{d}{dt} g^\sharp(a,d) = -2 d^\sharp(a,d) = \partial B^\sharp(A,D)\,. \tag{37}$$

If we denote by $D = \Phi_t^*(d)$, with $2d = \widehat{\nabla} v_t + (\widehat{\nabla} v_t)^{\mathrm{T}}$, then for its representation $\mathbf{D}$ by Remark-2

$$\mathbf{D} := \Phi^*(\mathbf{d}) \;=\; \Phi^*(\mathbf{g}^\sharp \mathbf{d}^\flat) = \mathbf{F}^{-1} \mathbf{g}^\sharp \mathbf{F}^{-*} \mathbf{F}^* \mathbf{d}^\flat \mathbf{F} = \mathbf{B}^\sharp \, \frac{1}{2} \partial \mathbf{C}^\flat = \frac{1}{2} \mathbf{B} \, \partial \mathbf{C} \tag{38}$$

$$\;=\; \Phi^*(\mathbf{d}^\sharp \mathbf{g}) = \mathbf{F}^{-1} \mathbf{d}^\sharp \mathbf{F}^{-*} \mathbf{F}^* \mathbf{g} \mathbf{F} = -\frac{1}{2} \partial \mathbf{B}^\sharp \, \mathbf{C}^\flat = -\frac{1}{2} \partial \mathbf{B} \, \mathbf{C}\,, \tag{39}$$

where $\partial \mathbf{C} = \partial(\mathbf{F}^{\mathrm{T}} \mathbf{F})$ and $\partial \mathbf{B} = \partial(\mathbf{F}^{-1} \mathbf{F}^{-\mathrm{T}})$.

Now a vector field v_t, for which $d^\flat(v_t) = \frac{1}{2} \, \mathcal{L}_{v_t} \, g = 0$, is called the *Killing vector field*. The most general form of time-dependent mappings $\Phi_t : \mathcal{E}^3 \to \mathcal{E}^3$ corresponding to Killing vector fields in $\mathcal{E}^3$ are *isometries*, i.e. time-dependent Euclidean transformations in $\mathcal{E}^3$ preserving distances within the body (see Marsden & Hughes (1993), p 99). That is to say, they represent a moving body in $\mathcal{E}^3$ without deformation.

Since the Cauchy-Green tensor field $C^\flat$ represents the convected (Riemannian) metric on the body $\mathcal{B}$ (in fact, the space $\mathcal{B}$ with $C^\flat = \Phi^*(g)$ is isometric to the space $\mathcal{S} = \Phi(\mathcal{B})$ with g, via the mapping Φ), it applies (see Simo et al. (1988))

$$\partial C^\flat = \mathcal{L}_{\mathbf{v}_t} \, C^\flat = 2 \left[(\widetilde{\nabla} \mathbf{v}_t)^\flat\right]_{sym} \tag{40}$$

$$\partial B^\sharp = \mathcal{L}_{\mathbf{v}_t} \, B^\sharp = 2 \left[(\widetilde{\nabla} \mathbf{v}_t)^\sharp\right]_{sym}\,, \tag{41}$$

where now $\widetilde{\nabla} = \Phi_t^*(\widehat{\nabla})$ denotes the covariant derivative associated with convective metric field $C^\flat$ in $\mathcal{B}$, and $\mathcal{L}_{\mathbf{v}_t} = \Phi_t^*(\mathcal{L}_{v_t})$. At the same time we have

$$\mathbf{a}_t = \left(\frac{\partial}{\partial t} + \widetilde{\nabla}_{\mathbf{v}_t}\right) \mathbf{v}_t \tag{42}$$

for convective acceleration field $\mathbf{a}_t$.

2.5. Dual stress and strain tensors, corresponding time derivatives

The various stress and strain tensors, and their objective time derivatives can be related to each other (Hill (1968), Haupt and Tsakmakis (1989)) via the *stress power density*:

$$\pi_t(x) \equiv \langle \sigma_t^\sharp, d_t^\flat \rangle_{T_x^*\mathcal{S}} = \langle \sigma_t^\flat, d_t^\sharp \rangle_{T_x\mathcal{S}} = \sigma_t : d_t \tag{43}$$

where σ_t is the mixed CAUCHY stress tensor, and d_t is the mixed rate-of-deformation tensor.

Hill's result is obtained by pulling-back the spatial picture to the referential configuration, so that the *referential stress power density* can be written:

$$\pi_t^{ref}(X) = J(\pi_t \circ \Phi)(X) = \left\{ \begin{array}{l} \langle P_t^\sharp, \partial E_t^\flat \rangle_{T_X^*\mathcal{B}} \\ \langle K_t^\flat, \partial H_t^\sharp \rangle_{T_X\mathcal{B}} \end{array} \right\} = \left\{ \begin{array}{l} P_t : \partial_t E_t \\ K_t : \partial_t H_t \end{array} \right. \tag{44}$$

Above, the following two relations were employed (see (34) and (35))

$$\Phi^*(d^\flat) = \partial E^\flat = \tfrac{1}{2}\partial C^\flat \qquad \Phi^*(d^\sharp) = -\partial H^\sharp = -\tfrac{1}{2}\partial B^\sharp$$

Now, $P^\sharp = \Phi^*(\tau^\sharp)$ is the SECOND PIOLA-KIRCHHOFF stress and $K^\flat = -\Phi^*(\tau^\flat)$ the NEGATIVE CONVECTED stress, where the mixed WEIGHTED CAUCHY (or KIRCHHOFF) stress tensor $\tau = J\sigma$. The Jacobian J (being scalar) is then the determinant of the tangent mapping transformation $J = \det(\partial\Phi/\partial X)\sqrt{\det(g)/\det(G)}$, where G and g denotes the metric on $\mathcal{B}$ and $\mathcal{S}$, respectively (i.e. the scalar product on $T_X\mathcal{B}$ and $T_x\mathcal{S}$, respectively).

Since $E^\flat = \Phi^*(e^\flat)$ and $e^\flat = \Phi_*(E^\flat)$, $H^\sharp = \Phi^*(h^\sharp)$ and $h^\sharp = \Phi_*(H^\sharp)$, then by pushing-forward the Hill's result back to spatial configuration Haupt and Tsakmakis obtained:

$$\pi_t^{ref}(X(x)) = \left\{ \begin{array}{ll} \langle \tau_t^\sharp, \mathcal{L}_{v_t} e_t^\flat \rangle_{T_x^*\mathcal{S}} & (\mathcal{L}_{v_t} e^\flat)_{ij} = \dot{e}_{ij} + (d-w)_i^l\, e_{lj} + e_{ik}\,(d+w)_j^k \\ \langle -\tau_t^\flat, \mathcal{L}_{v_t} h_t^\sharp \rangle_{T_x\mathcal{S}} & (\mathcal{L}_{v_t} h^\sharp)^{ij} = \dot{h}^{ij} - (d+w)_l^i\, h^{lj} - h^{ik}\,(d-w)_k^j \end{array} \right. \tag{45}$$

$\mathcal{L}_{v_t} = \Phi_* \circ \partial \circ \Phi^*$ is the Lie derivative, and w is the vorticity. This *time derivative* (known in mechanical literature as the OLDROYD time derivative), obtained from material derivative exactly the same way as the corresponding dual stress and strain tensors, is naturally *objective*.

T-$\mathcal{B}$:	$C^\flat = \Phi^*(g)$	the RIGHT CAUCHY-GREEN *deformation tensor*
	$E^\flat = \frac{1}{2}(C^\flat - G)$	the GREEN-ST.VENANT *strain tensor*
	$P^\sharp = \Phi^*(\tau^\sharp)$	the SECOND PIOLA-KIRCHHOFF *stress tensor*
T-$\mathcal{S}$:	$c^\flat = \Phi_*(G)$	the ALMANSI *deformation tensor*
	$e^\flat = \frac{1}{2}(g - c^\flat)$	the ALMANSI-HAMEL *strain tensor*
	$\tau^\sharp$	the (contravariant) WEIGHTED CAUCHY *stress tensor*
C-$\mathcal{B}$:	$B^\sharp = \Phi^*(g^\sharp)$	the PIOLA *deformation tensor*
	$H^\sharp = \frac{1}{2}(B^\sharp - G^\sharp)$	the PIOLA *strain tensor*
	$K^\flat = -\Phi^*(\tau^\flat)$	the NEGATIVE CONVECTED *stress tensor*
C-$\mathcal{S}$:	$b^\sharp = \Phi_*(G^\sharp)$	the LEFT CAUCHY-GREEN *deformation tensor*
	$h^\sharp = \frac{1}{2}(g^\sharp - b^\sharp)$	the FINGER *strain tensor*
	$-\tau^\flat$	the (covariant) WEIGHTED CAUCHY *stress tensor*

Above, *T* stands for tangent, *C* for cotangent spaces. The corresponding time derivative in $\mathcal{B}$ is ∂_t, in $\mathcal{S}$ the Lie derivative $\mathcal{L}_{v_t}$.

2.6. Summary

In order to emphasize the actual mechanical content of this overview let us conclude by the summary: **(0)** *Instead of mixed deformation tensor fields, it is geometrically more natural to think of their covariant, or possibly also contravariant, form.* **(1)** *Deformation of a body is completely characterized by a Riemannian metric – from referential configuration point of view by the covariant field of right Cauchy-Green deformation tensors* $C^\flat$*, from actual configuration viewpoint by the covariant Almansi deformation tensor field* $c^\flat$*.* **(2)** *The contravariant field of Piola deformation tensors* $B^\sharp$ *or the contravariant field of left Cauchy-Green deformation tensors* $b^\sharp$ *express exactly the same geometry but in rather less illustrative representation.* **(3)** *A deformation process can be represented by a trajectory* $C^\flat\colon I \to \mathcal{M}$ *in the space* $\mathcal{M} = \mathrm{Met}(\mathcal{B})$ *of all Riemannian metrics on the reference configuration* $\mathcal{B}$*. Within the space* $\mathcal{M}$*, a Riemannian metric* $C^\flat_t$ *at a time* t *is a point. Time derivative* $\partial C^\flat_t$ *here is a vector, which corresponds to the covariant rate-of-deformation field* $d^\flat_t$ *in actual configuration, via pull-back and push-forward transformations.* **(4)** *The strain and stress tensors form the so-called dual pairs with the corresponding time derivative.* **(5)** *In the next section, we prove moreover that the second Piola-Kirchhoff stress field is a covector.*

3. Geometry of the space $\mathcal{M}$ and the deformation process

We have concluded the preceding section by the summary, which provides us with the starting point for our next analysis of the deformation process:

STARTING POINT: *From the viewpoint of finite deformations, a deformation process can be represented by a trajectory* $C^\flat\colon I \to \mathcal{M}$ *in the space* $\mathcal{M} = Met(\mathcal{B})$ *of all Riemannian metrics on the reference configuration* $\mathcal{B}$*. If the initial configuration is unstrained, with an initial condition* $C^\flat_0 = G$*. Tanget vectors* $\partial C^\flat_t$ *correspond, via pull-back and push-forward transformations, to the covariant rate-of-deformation field* $d^\flat_t$ *in actual configuration.*

From the mathematical point of view, the space $\mathcal{M}$ forms an infinite-dimensional manifold. In this section we show that it may be given a Riemannian metric, i.e. a geometry, to become the Riemannian infinite-dimensional manifold of Riemannian metrics. Moreover, its geometry factorises into identical geometry of individual spaces $\mathcal{M}_X$, made up of all metric tensors at a point $X \in \mathcal{B}$ – that is $\mathcal{M}_X = Sym^+(n)$ the space of symmetric positive-definite matrices $Sym^+(n)$. Considering $Sym^+(n)$ as the Riemannian manifold rather than a vector space enables us to analyse in a geometrically consistent way a deformation process by means of geometrical tools of the Riemannian geometry. This approach to mechanics of continua was initiated by Rougée (1997), and further modified by Fiala (2004), (2007), where details of following propositions can be found

First we show that a deformation process within small deformations is formed by a trajectory in a linear vector space – the tangent space $T_{C^\flat}\mathcal{M}$ to the manifold $\mathcal{M}$ at a point $C^\flat$, representing the initial deformation state of the body.

PROPOSITION 1: *Within small deformations, a deformation process superposed on the initaily strained body, and chracterized by a deformation field* $C^\flat_0$*, is represented by a trajectory in a linear vector space – the tangent space* $T_{C^\flat_0}\mathcal{M}$ *to the manifold* $\mathcal{M}$ *at the point* $C^\flat_0$*.*

Proof. Let now $\Psi : I \times \Phi_0(\mathcal{B}) \to \mathcal{E}^3$ be a deformation process starting from the deformed state $\Phi_0(\mathcal{B})$, characterized by the right Cauchy-Green deformation field $C_0^\flat = \Phi_0^*(G_0)$, where G_0 is actual metric field in the intermediate configuration $\Phi_0(\mathcal{B})$, occupied by body $\mathcal{B}$ after deformation Φ_0. From the time-dependent diffeomorphism Ψ_t one obtains the deformation rate field $\partial C_t^\flat = 2\,(\Phi_0^* \circ \Psi_t^*)\,(d^\flat)$ and the Euler velocity field $v_t = \partial(\Psi_t \circ \Phi_0)\,(X)|_X$, all related together via (see (30), (34))

$$\Phi_{0*}(\partial C_t^\flat) = 2\Psi_t^*\big(d^\flat(v_t)\big) = \Psi_t^*(\mathcal{L}_{v_t}\, g) = 2\Psi_t^*\Big((\widehat{\nabla} v_t^\flat)_{sym}\Big)\,. \tag{46}$$

Since for the tensor field of small deformation: $e(u_t)^i_j = [(\widehat{\nabla} u_t)_{sym}]^i_j = \frac{1}{2}(u_t^i|_j + u_{t\,j}|^i)$, we get $d(v_t) = \partial e(u_t)$, where $v_t = \partial u_t$. Now, since for small deformations a diffeomorphism Ψ_t acts as the identity mapping, i.e. $x = \Psi_{Id}(\Phi_0(X)) \approx \Phi_0(X) = X_0$, the deformation gradient $T\Psi_t \equiv I + Tu_t \approx I$. For transformations of vectors and covectors we thus obtain $v = T\Psi_t(V_0) \approx V_0$ and $A_0 = T\Psi_t^*(a) \approx a$, which means that the concept of small deformations identifies tensor spaces in intermediate and actual configurations. In particular, the metric tensor fields are equal $g \approx G_0$, (besides, the Lie derivative is replaced by the simple time derivative $\mathcal{L}_{v_t} = \Phi_* \circ \partial \circ \Phi^* \approx \partial$.) Now, an infinitesimal variation $u(X_0)$ around the identity mapping $\Psi_{Id}(X_0) = x \approx X_0$ at a point $x = \Psi_t(X_0)$ (i.e. linearization of the mapping Ψ_t in other words) enters the theory of small deformations via a variation of the metric $\Psi_t^*(\mathcal{L}_{v_t}\, g) \approx \mathcal{L}_v\, g = 2e^\flat(v)$. Then from the relation $\Phi_{0*}(\partial C_t^\flat) = 2\Psi_t^*\big(d^\flat\big)$ for an increment within small deformation one obtains $\Phi_{0*}(\partial C_t^\flat) \approx 2d^\flat(v)$, i.e. $\partial C_t^\flat \approx 2\Phi_0^*(e^\flat(v))$, and the proposition follows.

PROPOSITION 2: *One can naturally introduce a Riemannian metric on $\mathcal{M}_X$ to become a manifold with a Riemannian geometry of $Sym^+(n) \cong GL^+(n)/SO(n)$, the space of symmetric positive-definite matrices. The space $Sym^+(n)$ is curved with constant negative curvature.*

Proof. A Riemannian metric on the space $\mathcal{M}_X$ will be introduced in terms of a scalar product on the vector space $T_{C^\flat}\mathcal{M}_X$, made up of tangent vectors $\partial C^\flat$ to all curves $C^\flat(t)$ passing through the point $C_X^\flat \in \mathcal{M}_X$. For this, the formula (34)

$$\partial C^\flat = 2\Phi_t^*(d^\flat)\,,$$

relating vectors $\partial C^\flat$ in $\mathcal{M}_X$ with the rate-of-deformation tensor fields $d^\flat$ over $\mathcal{S}$, becomes crucial. Since the scalar product $g(u,v) = g_{ij}u^iv^j = u_iv^i$ of two vectors from $\mathcal{S}$ naturally extends to the scalar product of symmetric covariant 2-tensors (see 82)

$$(g^\sharp \otimes g^\sharp)(d^\flat, h^\flat) = g^{ik}g^{jl}\, d_{ij}\, h_{kl} = d_{ij}\, h^{ij}\,, \tag{47}$$

its corresponding counterpart in reference configuration $\mathcal{B}$, thanks to (34), has the following form

$$\Gamma_X(D^\flat, H^\flat) = B^{ik}B^{jl}\, D_{ij}\, H_{kl} = B^i_k D^k_l B^l_n H^n_i \equiv \mathrm{tr}\,(\mathbf{C}^{-1}\mathbf{D}\mathbf{C}^{-1}\mathbf{H}) \tag{48}$$

and it represents the scalar product of two symmetric covariant tensors of second order $D^\flat = \Phi_t^*(d^\flat)$, $H^\flat = \Phi_t^*(h^\flat)$. For more about geometry of $Sym^+(n)$ see Bhatia (2007) and Fiala (2007), where further references are included.

PROPOSITION 3: *A Riemannian metric on $\mathcal{M}$ is given by* (Fiala (2007))

$$\Gamma_{C^\flat}(D^\flat, H^\flat) := \int_{\mathcal{B}} \Gamma_X(D^\flat, H^\flat)\, dm = \int_{\mathcal{B}} B^{ik}B^{jl}\, D_{ij}H_{kl}\, dm\,, \tag{49}$$

where dm *denotes the mass element.*

PROPOSITION 4: *The convective stress field* $K^\flat = \Phi_t^*(J\sigma^\flat)$ *is a* ***vector****, and the second Piola-Kirchhoff stress field* $P^\sharp = \Phi_t^*(J\sigma^\sharp)$ *a* ***covector****.*

Proof. In fact, the power of internal forces (*stress power*) can be written in several ways:

$$
\begin{aligned}
\frac{\delta E_i}{\delta t} &:= \int_{\mathcal{S}} (g^\sharp \otimes g^\sharp)(\sigma^\flat, d^\flat)\, dv \equiv \int_{\mathcal{S}} g^{ik} g^{jl} \sigma_{kl}\, d_{ij}\; dv = \\
&= \int_{\mathcal{B}} B_t^{ik} B_t^{jl} K_{kl}\, \tfrac{1}{2}\, \partial C_{t\,ij}\, dV = \Gamma_{C_t^\flat}\left(\frac{1}{\rho_{\mathcal{B}}} K^\flat, \frac{1}{2}\, \partial C_t^\flat\right) \qquad (50) \\
&:= \int_{\mathcal{S}} \langle \sigma^\sharp, d^\flat \rangle_{T_x \mathcal{S}}\, dv \equiv \int_{\mathcal{S}} \sigma^{ij} d_{ij}\; dv = \\
&= \int_{\mathcal{B}} P^{ij}\, \tfrac{1}{2}\, \partial C_{t\,ij}\, dV = \int_{\mathcal{B}} \langle P^\sharp, \tfrac{1}{2}\, \partial C_t^\flat \rangle_{T_X \mathcal{B}}\, dV = \left\langle \frac{1}{\rho_{\mathcal{B}}} P^\sharp, \frac{1}{2}\, \partial C_t^\flat \right\rangle_{T_{C_t^\flat} \mathcal{M}} \qquad (51)
\end{aligned}
$$

where we made use of relations $\frac{1}{2}\partial C_t^\flat = \Phi_t^*(d^\flat)$, $J dV = \Phi_t^*(dv)$, and $dV = G^{\frac{1}{2}} dX$ for volume element. The symbol σ as usual stands for the Cauchy stress field. The quantities $\frac{1}{\rho_{\mathcal{B}}} K^\flat$, resp $\frac{1}{\rho_{\mathcal{B}}} P^\sharp$ are the convective stress, resp the second Piola-Kirchhoff stress fields related to the mass, instead of the volume element.

PROPOSITION 5: *If the material is* ***hyperelastic****, then*

$$
\frac{1}{\rho_{\mathcal{B}}} P^\sharp = 2 d^{\mathcal{M}} E_i \quad \text{i.e.} \quad (P^\sharp)^{ij}(X) = 2 \frac{\partial \varepsilon}{\partial C_{ij}}(X) = 2 \rho_{\mathcal{B}} \frac{\partial e}{\partial C_{ij}}(X)\,, \qquad (52)
$$

where $d^{\mathcal{M}}$ *stands for exterior derivative/diferential corresponding to* $\mathcal{M}$ *(see Frankel (1997), Marsden & Hughes (1993), or Schutz (1999) for example), and* $E_i = \int_{\mathcal{B}} \varepsilon\, dV = \int_{\mathcal{B}} \rho_{\mathcal{B}}\, e\, dV$*.*

Proof. A change in strain energy ΔE_i in the body $\mathcal{B}$ after deformation process $C^\flat : \langle t_0, t \rangle \to \mathcal{M}$ can be written down in terms of a curve integral

$$
\Delta E_i = \int_{\langle t_0, t \rangle} \left\langle \frac{1}{\rho_{\mathcal{B}}} P^\sharp, \frac{1}{2} \partial C_t^\flat \right\rangle_{T_{C_t^\flat} \mathcal{M}} dt \equiv \int_{C_t^\flat} \frac{1}{2\rho_{\mathcal{B}}} P^\sharp. \qquad (53)
$$

If the strain energy ΔE_i does not depend on the integration path, then the strain energy as a function on $\mathcal{M}$ forms a *potential*, and so $\frac{1}{\rho_{\mathcal{B}}} P^\sharp = 2 d^{\mathcal{M}} E_i$. If we further express E_i as

$$
E_i = \int_{\mathcal{B}} \varepsilon\, dV = \int_{\mathcal{B}} \rho_{\mathcal{B}}\, e\, dV, \qquad (54)
$$

then thanks to the local character of the metric Γ, i.e. the fact that Γ induces on each tangent space $T_X \mathcal{B}$ a scalar product, we have

$$
\begin{aligned}
\frac{1}{\rho_{\mathcal{B}}} P^\sharp &= 2 d^{\mathcal{M}} E_i = 2 \frac{\partial E_i}{\partial C_{ij}}(C^\flat)\, dC_{ij} \qquad (55) \\
&= 2 \int_{\mathcal{B}} \left(\frac{\partial \varepsilon}{\partial C_{ij}}\, dC_{ij} \right)(X)\, dV = 2 \int_{\mathcal{B}} \left(\frac{\partial e}{\partial C_{ij}}\, dC_{ij} \right)(X)\, dm\,,
\end{aligned}
$$

which results into the following well-known local formula

$$(P^\sharp)^{ij}(X) \equiv (P^\sharp(C^\flat(X)))^{ij} = 2\frac{\partial \varepsilon}{\partial C_{ij}}(X) = 2\rho_{\mathcal{B}}\frac{\partial e}{\partial C_{ij}}(X)\,. \tag{56}$$

PROPOSITION 6: *The stress rate is given by the Zaremba-Jaumann objective time derivative.*

Proof. We can naturally identify the ***time derivative*** of the time-dependent covector or vector field Θ, and its corresponding 2-tensor field $\theta = \Phi_{t*}(\Theta)$ on $\mathcal{E}^3$, with the time derivative based on covariant derivative in $\mathcal{M}$ with respect to the deformation rate $\partial C_t^\flat$ (cf. (25))

$$\frac{D}{dt}\Theta := \frac{\partial\Theta}{\partial t} + \nabla_{\partial C_t^\flat}\Theta \qquad \text{on the space } \mathcal{M} \tag{57}$$

$$\frac{D}{dt}\theta := \Phi_{t*}\left(\frac{D}{dt}\Theta\right) \qquad \text{on the space } \mathcal{S}. \tag{58}$$

The symbol ∇ stands for the covariant derivative associated with the metric Γ on the manifold $\mathcal{M}$, which automatically guarantees an objectivity of the time derivative induced.

For a time-dependent *vector field* U along a trajectory $C_t^\flat$ in $\mathcal{M}$, we thus obtain

$$\left(\frac{D}{dt}U\right)_{mp} \equiv \left(\frac{\partial}{\partial t}U + \nabla_{\partial C_t^\flat}U\right)_{mp} = \dot{U}_{mp} - \frac{1}{2}\left\{(\partial C_t)_{ma}B_t^{ab}U_{bp} + U_{ma}B_t^{ab}(\partial C_t)_{bp}\right\}\,, \tag{59}$$

for the *covector field* Ω, we analogically get

$$\left(\frac{D}{dt}\Omega\right)^{ij} \equiv \left(\frac{\partial}{\partial t}\Omega + \nabla_{\partial C_t^\flat}\Omega\right)^{ij} = \dot{\Omega}^{ij} + \frac{1}{2}\left\{\Omega^{ia}(\partial C_t)_{ab}B_t^{bj} + B_t^{ia}(\partial C_t)_{ab}\Omega^{bj}\right\}. \tag{60}$$

Making use of (58) with the help of expression (22) we obtain the time derivatives of corresponding second order tensor fields on $\mathcal{S}$ – the *Zaremba-Jaumann derivative*. For the *symmetric covariant tensor field of second order* u on the actual configuration $\mathcal{S}$, represented along a trajectory $C_t^\flat$ in $\mathcal{M}$ by a *vector field* $U = \Phi_t^*(u)$, such as the covariant Kirchhoff stress $\tau^\flat = J\sigma^\flat$, we obtain

$$\left(\frac{D}{dt}u\right)_{mp} = \dot{u}_{mp} - w_m{}^k u_{kp} + u_{mk}w^k{}_p \equiv \left(\mathring{u}^{\mathbf{ZJ}}\right)_{mp}. \tag{61}$$

Similarly, for the *symmetric contravariant tensor field of second order* ω on the actual configuration $\mathcal{S}$, represented along a trajectory $C_t^\flat$ in $\mathcal{M}$ as a *covector field* $\Omega = \Phi_t^*(\omega)$, such as the contravariant Kirchhoff stress $\tau^\sharp = J\sigma^\sharp$, is expressed by

$$\left(\frac{D}{dt}\omega\right)^{ij} = \dot{\omega}^{ij} - w^i{}_k\,\omega^{kj} + \omega^{ik}w_k{}^j \equiv \left(\mathring{\omega}^{\mathbf{ZJ}}\right)^{ij}, \tag{62}$$

where $2w_{ij} = v_i|_j - v_j|_i$ is the *vorticity*. In particular, for the ***time derivative of the Cauchy stress field*** $\sigma^\sharp$ during deformation process we obtain

$$\frac{D}{dt}\sigma^\sharp = \frac{1}{J}(\mathring{\tau}^\sharp)^{\mathbf{ZJ}} - (\mathbf{tr}_g d^\flat)\,\sigma^\sharp. \tag{63}$$

PROPOSITION 7: *The logarithmic strain field is a vector, and it corresponds to a geodesic line (straight line) connecting undeformed state with deformed state.*

Proof. A geodesic stands for a generalization of the straight line – locally the shortest connecting line between two points. It can also be characterized, as in the Euclidean space alike, by the fact that its tangent vectors form a parallel vector field, and so for our case it satisfies the equation (cf. (59))

$$\nabla_{\partial C_t^\flat}\, \partial C_t^\flat = 0\,, \tag{64}$$

resulting in

$$\partial C_{ij}(t) = C_{ik}(t) 2D_j^k \tag{65}$$

Its solutions are geodesics parametrized by the constant, mixed tensor fields $2D_j^k$

$$C_{ij}(t) = C_{ik}(0)\,(\exp 2t\mathbf{D})_j^k\ , \tag{66}$$

where the exponential mapping means the matrix exponential (as usual in this paper by bold capital letters we denote matrix fields made up from components of tensor fields, or their representations as linear mappings).

If the geodesic is prescribed by an initial point $C_0^\flat$ and an initial velocity $\partial C_0^\flat$, the matrix field $\mathbf{D}$ reads

$$2\mathbf{D} = \mathbf{B}_0^\sharp\, \partial \mathbf{C}_0^\flat = \mathbf{B}_t^\sharp\, \partial \mathbf{C}_t^\flat\,, \tag{67}$$

if it is determined by two points $C_0^\flat$ and $C_\tau^\flat$, then

$$2\mathbf{D} = \log\,(\mathbf{B}_0^\sharp \mathbf{C}_\tau^\flat)/\tau = \log\,(\mathbf{B}_0^\sharp \mathbf{C}_1^\flat)\,. \tag{68}$$

Recall that $\mathbf{B} = \mathbf{C}^{-1}$. The geodesic (66), expressed in terms of mixed 2-tensor field equals to $\mathbf{C}_t = \mathbf{G}^{-1}\mathbf{C}_t^\flat$, and so

$$\mathbf{C}_t = \mathbf{C}_0 \exp(t\mathbf{B}_0\, \partial \mathbf{C}_0) = \mathbf{C}_0 \exp(t\mathbf{C}_0^{-1} \partial \mathbf{C}_0)\,. \tag{69}$$

If the initial point is $\mathbf{I}$ then $\mathbf{C}_t = \exp(t\,\partial \mathbf{C}_0)$, and so $2\mathbf{D} = \partial \mathbf{C}_0 = \log \mathbf{C}_1$, and the proposition follows. Moreover, we have proved another two propositions:

PROPOSITION 8: *A* ***generalization of the logarithmic strain*** *field, due to the foregoing interpretation as a vector corresponding to a geodesic line connecting two deformed states $C_0^\flat$ and $C^\flat$, is given by*

$$C^\flat \longmapsto \log_{C_0^\flat}(C^\flat) := H_0^\flat \cong \tfrac{1}{2}\mathbf{C}_0^\flat \log\,(\mathbf{B}_0^\sharp \mathbf{C}^\flat)\,. \tag{70}$$

PROPOSITION 9: *Resulting deformation from adding an increment $H^\flat$ to a deformation $C^\flat$ is given by*

$$H^\flat \longmapsto \exp_{C_0^\flat}(H^\flat) := C_{H^\flat}^\flat(1) \cong \mathbf{C}_0^\flat \exp(2\mathbf{B}_0^\sharp\, \mathbf{H}^\flat)\,. \tag{71}$$

4. Incremental approach to finite deformations

Incremental approach to finite deformations in fact reduces to a general theory of small deformations that are step by step superposed on a known finite deformation to obtain a new, updated finite deformation.

Let us briefly return to deformation process via diffeomorphisms. The space of all diffeomorphisms of a body can be given a natural Riemannian geometry such that the geodesic in this space, determined by a known diffeomorphism Φ_t (as the starting point) and by a vector field v (as the velocity vector), is given by line segments determined at each point by individual vectors of the vector field (see Ebin & Marsden (1970), theorem 9.1 (iii)). That means, the updated deformation at time $t + \Delta t$, after superposed infinitesimal incremental deformation v to a deformation at time t, is given by

$$\Phi_{t+\Delta t}(X) \equiv (\Psi_{\Delta t} \circ \Phi_t)(X) = \Phi_t(X) + \Delta t\, (v \circ \Phi_t)(X)\,, \tag{72}$$

which is what one would anticipate.

But as the preceding section suggests, situation is completely different with the right Cauchy-Green deformation tensor $C^\flat$ due to the curvature of the space $\mathcal{M}$. In fact, from the already established vector field v superposed on the deformation $C_t^\flat \in \mathcal{M}$, the updated deformation $C_{t+\Delta t}^\flat \in \mathcal{M}$ should be expressed in terms of the exponential mapping (cf. (69))

$$\begin{aligned} \mathbf{C}_{t+\Delta t} &= \mathbf{C}_t \exp\left[\Delta t \mathbf{C}_t^{-1} \partial \mathbf{C}_t(v)\right] \\ &= \mathbf{C}_t + \Delta t\, \partial \mathbf{C}_t(v) + \tfrac{1}{2!}(\Delta t)^2\, \partial \mathbf{C}_t(v) \mathbf{C}\, \partial \mathbf{C}_t(v) + \ldots\,, \end{aligned} \tag{73}$$

where the deformation rate $\partial C_t^\flat(v) \in T_{C_t^\flat}\mathcal{M}$ is given by (see (46) and Proposition 1)

$$\partial C_t^\flat(v) = 2\Phi_t^*\left(d^\flat(v)\right) = \Phi_t^*\left(\mathcal{L}_v\, g\right) = 2\Phi_t^*\left((\widehat{\nabla} v^\flat)_{sym}\right) = (\widetilde{\nabla}\mathbf{v}^\flat)_{sym}\,. \tag{74}$$

Again $\widetilde{\nabla} = \Phi_t^*(\widehat{\nabla})$ denotes the covariant derivative associated with convective metric field $C_t^\flat$ in $\mathcal{B}$ (see (20), (40)), and $\mathbf{v} = \Phi_t^*(v)$ denotes the convective velocity field (see diag. after (18)).

For the corresponding stress (cf. (56) and (60))

$$P_{t+\Delta t}^{ij} = \left.\frac{\partial \epsilon}{\partial C_{ij}}\right|_{t+\Delta t} \qquad \text{and} \qquad \frac{D}{dt}\, P_t^\sharp \neq \lim_{\Delta t \to 0} \frac{P_{t+\Delta t}^\sharp - P_t^\sharp}{\Delta t} \equiv \frac{d}{dt}\, P_t^\sharp\,. \tag{75}$$

Now we can compare from this geometric point of view the incremental approach of Green & Zerna with that of Biot. Green & Zerna established the strain (deformation) increment in the same form as (74), but for the updated deformation they employ just the first too terms of (73). As for the stresses (75), they work with the simple time derivative, which does not result in the Zaremba-Jaumann time derivative in actual configuration. On the other hand, Biot rightly deduced the Zaremba-Jaumann derivative from mechanical considerations, but for the updated deformation he still employs only the first too terms of (73).

5. Conclusion

Elaborating the approach to the mechanics of continua in the framework of the infinite-dimensional Riemannian geometry, we have proved that the deformation tensor field and its corresponding logarithmic strain field are two totally different quantities – a point and a vector in the infinite-dimensional Riemannian manifold of Riemannian metrics, made up of the right Cauchy-Green deformation fields. In this space, the logarithmic strain field forms a vector determining locally the shortest connecting line – the geodesic – between the field of identity tensors representing the undeformed state and a deformation field representing a deformed state. Such an interpretation enables to introduce the logarithmic strain more generally for the case where the initial state may be any deformed state (see (70)). This brings a new idea to the problems related to the initial states that are not stress-free (see Bruhns et all. (2002)).

As for the differences between Green & Zerna and Biot, we can conclude that Biot actually considers the stress space as if the underlying space $\mathcal{M}$ were curved, while the space $\mathcal{M}$ itself as Euclidean vector space. Green & Zerna in both cases stem only from vectorial nature of the space $\mathcal{M}$ of deformation tensor field.

Finally, I would like to point out other, very close applications of the geometry of the space of symmetric positive-definite matrices. Namely, that of averaging symmetric positive-definite tensors applied in elasticity Moakher (2006) and finding the closest elastic tensor of arbitrary symmetry to an elasticity tensor of lower symmetry Moakher & Norris (2006), with underlying mathematical exposition in Moakher (2005), and of using the so-called Riemannian elasticity in image analysis Pennec et all. (2005) and Pennec (2006), again with mathematical exposition in Pennec et all. (2006).

Appendix – Mathematical preliminaries

Let a continuous body $\mathcal{B}$ occupy a region of the three-dimensional Euclidean point space $\mathbb{E}^3$. Now, from the technical point of view it is convenient to consider the Euclidean space $\mathbb{E}^3$ as a ***Riemannian manifold*** $\mathcal{E}^3$. For our purposes it suffices to characterize the Riemannian manifold as a set of points, with no privileged coordinate system, endowed with a Riemannian metric, which enters the manifold via a scalar product on tangent spaces. For more of geometrical tools relevant to continuum mechanics, see Frankel (1997), Schutz (1999), Dodson & Poston (1997) or Abraham et al. (1988).

• The *tangent space* $T_X\mathcal{B}$ is a linearized, infinitesimal neighbourhood of a point $X \in \mathcal{B}$. It is a linear, finite-dimensional real vector space of all "infinitesimal material line elements" represented by *vectors*

$$u = c'(0), \tag{76}$$

tangent at the point $X = c(0)$ to curves $c(t)$ in $\mathcal{B}$.

• The *cotangent space* $T^*_X\mathcal{B}$ is again a linear, finite-dimensional real vector space, now the space of all linearized at the point X functions on $\mathcal{B}$, called *covectors* :

$$\lim_{t\to 0} \frac{f[\,c(t)] - f[X]}{t} = \langle a, u\rangle_{T_X\mathcal{B}}, \tag{77}$$

where $c'(0) = u$. From the viepoint of analysis, a covector corresponding to a function is given by its Frchet derivative. On the other hand, from the geometric point of view, covectors are linear mappings a from the tangent space $u \in T_X\mathcal{B}$ to real numbers $\langle a, u\rangle_{T_X\mathcal{B}} \in \mathbb{R}$, and so the cotangent space $T^*_X\mathcal{B}$ is the *dual* to its tangent space.

• Let a triple of numbers $\{X^i\}$ denotes *coordinates* of a point X. Then the induced canonical covariant basis $\{\mathrm{G}_i\}$ in the tangent space $T_X\mathcal{B}$, reads $\mathrm{G}_i := \partial/\partial X^i \equiv \partial X^i$, whereas the canonical contravariant basis $\{\mathrm{G}^i\}$ in the cotangent space $\mathrm{G}^i := dX^i$, and of course

$$\langle\, \mathrm{G}^i, \mathrm{G}_j\rangle_{T_X\mathcal{B}} = \delta^i_j\,. \tag{78}$$

• Having defined tensors on a manifold, we can introduce the key notion of Riemannian geometry, namely the ***metric***. It is a symmetric positive-definite covariant 2-tensor G defining the *scalar product* $G(u, v)$ of two vectors $u, v \in T_X\mathcal{B}$ with the same footpoint X, and thus *local geometry in an infinitesimal neighbourhood* of the point X

$$G(u, v) = G_{ij} u^i v^j = u_j \, v^j \equiv \langle u^\flat, v \rangle_{T_X\mathcal{B}} \, , \tag{79}$$

where $G_{ij} = G(\mathbf{G}_i, \mathbf{G}_j)$.

Moreover, the metric G makes it possible to introduce a mapping $\mathbf{G} : T_X\mathcal{B} \to T_X^*\mathcal{B}$ via the relation

$$G(u, v) = \langle \mathbf{G} u, v \rangle_{T_X\mathcal{B}} \, , \tag{80}$$

so that one can assign to a vector u an associated covector $u^\flat = \mathbf{G}u$, and conversely to a covector a an associated vector $a^\sharp = \mathbf{G}^{-1}a$. Of course $u = u^\sharp$ and $a = a^\flat$. The covectors are thus related to "infinitesimal material surface elements" represented by gradients at point X to surfaces, defined by the level sets of functions "$f(X) = $constant". The so-called *associated tensors* $t^\flat$, $t^\sharp$ to a mixed tensor t are (2-0)- and (0-2)-tensors respectively, defined by extending the mapping $\mathbf{G}$ to 2-tensors. These operations correspond to *raising and lowering indexes* of components of tensors, in classical approach.

The scalar product of vectors naturally extends to *scalar product of tensors of arbitrary order*. In particular, for covectors we thus obtain

$$G^\sharp(a, b) = G^{ij} a_i b_j = a_i b^i \equiv \langle a, b^\sharp \rangle_{T_X\mathcal{B}} \, , \tag{81}$$

for covariant 2-tensors

$$\left(G^\sharp \otimes G^\sharp\right)(d, h) = G^{ik} G^{jl} d_{ij} \, h_{kl} = d_{ij} \, h^{ij} \equiv d : h \, , \tag{82}$$

where G^{ij} is defined by $G_{ij} \, G^{jk} = \delta_i^k$, and naturally $G^{ij} = G^\sharp(\mathbf{G}^i, \mathbf{G}^j)$.

• A mapping $\Phi : \mathcal{B} \to \mathcal{S}$ between two manifolds induces the *tangent mapping* $T\Phi$ between corresponding tangent spaces $T\Phi : T_X\mathcal{B} \to T_x\mathcal{S}$, where $x = \Phi(X)$. In mechanics, it is usualy denoted as **F**, and incorrectly called the "deformation gradient" although it is not a gradient at all. To the tangent mapping $T\Phi$ one can assign its *dual mapping* $(T\Phi)^* : T_x^*\mathcal{S} \to T_X^*\mathcal{B}$, and its *transposed mapping* $(T\Phi)^{\mathrm{T}} : T_x\mathcal{S} \to T_X\mathcal{B}$ (see Remark-3).

• The tangent mapping with its dual defines then ***push-forward*** Φ_* and ***pull-back*** Φ^* operations between corresponding spaces of tensors. If the mapping Φ is a diffeomorphism, these then in a simple way couple the description of deformation and stress state in the reference and actual configurations: In fact, the *description of the motion in the reference (actual) picture is obtained by pull-back (push-forward) of the actual (reference) picture* (see Remark-4).

Remark-1 : Unlike the classical approach, making use of the dual space $\mathbb{U}^*$ of a vector space $\mathbb{U}$ enables us to define tensors more clearly, and distinguish between vectors and covectors, contravariant and covariant tensors, being considered here as different objects. Now, p-contravariant, q-covariant (p-q)-tensors are elements of the sets $T^{(p,q)} = T^{(p,0)} \otimes T^{(0,q)} = \mathbb{U} \otimes \cdots \otimes \mathbb{U} \otimes \mathbb{U}^* \otimes \cdots \otimes \mathbb{U}^*$, defining $T^{(0,0)} = \mathbb{R}$. There is another equivalent definition of (p-q)-tensor, as an element of the space of all polylinear functions $\mathrm{Lin}_{p+q}(\mathbb{U}^{*p} \times \mathbb{U}^q, \mathbb{R})$ to real numbers $\mathbb{R}$. For 2-tensor, see Remark-2.

Remark-2 : Of particular importance in mechanics of continua are 2-order tensors: (0-2), (1-1), and (2-0)-tensors elements of the spaces $\mathrm{Lin}_2(\mathbb{U}^2,\mathbb{R})$, $\mathrm{Lin}_2(\mathbb{U}^{*2},\mathbb{R})$, and $\mathrm{Lin}_2(\mathbb{U}^*\times\mathbb{U},\mathbb{R})$, respectively (i.e. covariant, mixed, and contravariant tensors). Because of $m(u,v) = \langle \mathbf{m}(v), u\rangle_{\mathbb{U}}$ (cf. (80)), note that the natural isomorphism between $m \in \mathrm{Lin}_2(\mathbb{U}^2,\mathbb{R})$ and $\mathbf{m} \in \mathrm{Lin}(\mathbb{U},\mathbb{U}^*)$ holds. Thus

$$\begin{aligned}
&\text{(0-2)}: && \mathbb{U}^*\otimes\mathbb{U}^* \simeq \mathrm{Lin}_2\left(\mathbb{U}^2,\mathbb{R}\right) && \simeq \mathrm{Lin}\left(\mathbb{U},\mathbb{U}^*\right) && \\
&\text{(1-1)}: && \mathbb{U}\otimes\mathbb{U}^* \simeq \mathrm{Lin}_2\left(\mathbb{U}^*\times\mathbb{U},\mathbb{R}\right) && \simeq \mathrm{Lin}\left(\mathbb{U},\mathbb{U}^{**}\right) && \simeq \mathrm{Lin}\left(\mathbb{U},\mathbb{U}\right) \\
&\text{(0-2)}: && \mathbb{U}\otimes\mathbb{U} \simeq \mathrm{Lin}_2\left(\mathbb{U}^{*2},\mathbb{R}\right) && \simeq \mathrm{Lin}\left(\mathbb{U}^*,\mathbb{U}^{**}\right) && \simeq \mathrm{Lin}\left(\mathbb{U}^*,\mathbb{U}\right)
\end{aligned} \tag{83}$$

Given a scalar product G on $\mathbb{U}$ (represented by $\mathbf{G} : \mathbb{U} \to \mathbb{U}^*$), then for (0-2), (1-1), and (2-0)-tensors **a**, **b**, and **c**, regarded as elements of $\mathrm{Lin}(\mathbb{U},\mathbb{U}^*)$, $\mathrm{Lin}(\mathbb{U},\mathbb{U})$, and $\mathrm{Lin}(\mathbb{U}^*,\mathbb{U})$ respectively, it holds

$$\begin{aligned}
&\mathbf{a}\left(\approx a \equiv a^\flat\right) \in \text{(0-2)}: && \mathbf{G}^{-1}\mathbf{a} && \in \text{(1-1)} \text{ and } && \mathbf{G}^{-1}\mathbf{a}\mathbf{G}^{-1}\left(\approx a^\sharp\right) \in \text{(2-0)} \\
&\mathbf{b} \qquad\qquad\quad \in \text{(1-1)}: && \mathbf{G}\mathbf{b}\left(\approx b^\flat\right) && \in \text{(0-2)} \text{ and } && \mathbf{b}\mathbf{G}^{-1}\left(\approx b^\sharp\right) \in \text{(2-0)} \\
&\mathbf{c}\left(\approx c \equiv c^\sharp\right) \in \text{(2-0)}: && \mathbf{c}\mathbf{G} && \in \text{(1-1)} \text{ and } && \mathbf{G}\mathbf{c}\mathbf{G}\left(\approx c^\flat\right) \in \text{(0-2)}
\end{aligned} \tag{84}$$

Remark-3 : To any linear mapping $\mathbf{L} : \mathbb{U} \to \mathbb{V}$ between two linear spaces, one can assign its *dual mapping* $\mathbf{L}^* : \mathbb{V}^* \to \mathbb{U}^*$ between their dual spaces, such that

$$\langle \mathbf{L}^* a, u\rangle_{\mathbb{U}} = \langle a, \mathbf{L}u\rangle_{\mathbb{V}}\,. \tag{85}$$

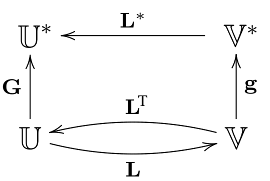
Moreover, if these spaces are endowed with scalar produts G on $\mathbb{U}$, and g on $\mathbb{V}$, one can also introduce the *transposed mapping* $\mathbf{L}^{\mathrm{T}} : \mathbb{V} \to \mathbb{U}$ defined by relation

$$G(\mathbf{L}^{\mathrm{T}} v, u) = g(v, \mathbf{L}u). \tag{86}$$

After combining this relation with $g(v_1,v_2) = \langle \mathbf{g}v_1, v_2\rangle_{\mathbb{V}}$ and $G(u_1,u_2) = \langle \mathbf{G}u_1, u_2\rangle_{\mathbb{U}}$, one obtains

$$\mathbf{L}^{\mathrm{T}} = \mathbf{G}^{-1}\mathbf{L}^*\mathbf{g} \tag{87}$$

and the folowing diagram applies



Remark-4:
For functions f on $\mathcal{B}$, and h on $\mathcal{S}$

$$\begin{aligned}
&\text{pull-back} && \Phi^*(h) = h \circ \Phi \\
&\text{push-forward} && \Phi_*(f) = f \circ \Phi^{-1},
\end{aligned} \tag{88}$$

for vectors $V \in T_X\mathcal{B}$, and $v \in T_x\mathcal{S}$

$$\begin{aligned}
&\text{pull-back} && \Phi^*(v) = (T\Phi)^{-1} \circ v \circ \Phi \\
&\text{push-forward} && \Phi_*(V) = T\Phi \circ V \circ \Phi^{-1},
\end{aligned} \tag{89}$$

for covectors $a \in T^*_X\mathcal{B}$, and $A \in T^*_x\mathcal{S}$

$$\begin{array}{ll} \text{pull-back} & \Phi^*(a) = (T\Phi)^* \circ a \circ \Phi \\ \text{push-forward} & \Phi_*(A) = (T\Phi)^{-*} \circ A \circ \Phi^{-1}, \end{array} \tag{90}$$

for general tensors Θ on $\mathcal{B}$, and θ on $\mathcal{S}$

$$\begin{array}{ll} \text{pull-back} & \Phi^*(\theta)(A, \ldots; V) = \theta\,(\Phi_*(A), \ldots; \Phi_*(V)) \\ \text{push-forward} & \Phi_*(\Theta)(a, \ldots; v) = \Theta\,(\Phi^*(a), \ldots; \Phi^*(v)). \end{array} \tag{91}$$

In the case of 2-order tensors in the representation by Remark 2, one obtains:

$$\begin{array}{l} \text{(0-2)}: \mathbf{a} \in \mathrm{Lin}\,(T_x\mathcal{S}, T^*_x\mathcal{S}) \to \mathbf{\Phi}^*(\mathbf{a}) = (T\Phi)^*\,\mathbf{a}\;T\Phi \quad \in \mathrm{Lin}\,(T_X\mathcal{B}, T^*_X\mathcal{B}) \\ \text{(1-1)}: \mathbf{b} \in \mathrm{Lin}\,(T_x\mathcal{S}, T_x\mathcal{S}) \to \mathbf{\Phi}^*(\mathbf{b}) = (T\Phi)^{-1}\,\mathbf{b}\;T\Phi \quad \in \mathrm{Lin}\,(T_X\mathcal{B}, T_X\mathcal{B}) \\ \text{(2-0)}: \mathbf{c} \in \mathrm{Lin}\,(T^*_x\mathcal{S}, T_x\mathcal{S}) \to \mathbf{\Phi}^*(\mathbf{c}) = (T\Phi)^{-1}\,\mathbf{c}\;(T\Phi)^{-*} \in \mathrm{Lin}\,(T^*_X\mathcal{B}, T_X\mathcal{B}) \end{array} \tag{92}$$

$$\begin{array}{l} \text{(0-2)}: \mathbf{A} \in \mathrm{Lin}\,(T_X\mathcal{B}, T^*_X\mathcal{B}) \to \mathbf{\Phi}_*(\mathbf{A}) = (T\Phi)^{-*}\mathbf{A}\;(T\Phi)^{-1} \in \mathrm{Lin}\,(T_x\mathcal{S}, T^*_x\mathcal{S}) \\ \text{(1-1)}: \mathbf{B} \in \mathrm{Lin}\,(T_X\mathcal{B}, T_X\mathcal{B}) \to \mathbf{\Phi}_*(\mathbf{B}) = T\Phi\;\mathbf{B}\;(T\Phi)^{-1} \quad \in \mathrm{Lin}\,(T_x\mathcal{S}, T_x\mathcal{S}) \\ \text{(2-0)}: \mathbf{C} \in \mathrm{Lin}\,(T^*_X\mathcal{B}, T_X\mathcal{B}) \to \mathbf{\Phi}_*(\mathbf{C}) = T\Phi\;\mathbf{C}\;(T\Phi)^* \quad \in \mathrm{Lin}\,(T^*_x\mathcal{S}, T_x\mathcal{S}) \end{array} \tag{93}$$

Of course, $\Phi_*(\Theta) = (\Phi^{-1})^*(\Theta) = (\Phi^*)^{-1}(\Theta)$, and vice versa.

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