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BOUNDARY CONDITIONS AND VIBRATION OF SLENDER BEAMS

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Summary: *The list of all possible boundary conditions corresponding to particular differential operator is much greater than usually presented in open literature. Gallery of various types of boundary conditions is presented for coupled bending-torsional vibration of a bar in compression and torsion. General condition to decide the selfadjointness is derived. Referenced cases are classified into selfadjoint or nonselfadjoint boundary conditions. Class of problems is distinguished, for which their selfadjointness guarantees the validity of the Euler method of stability analysis. For nonselfadjoint boundary conditions the stability analysis has to be carried out as dynamical problem even when the bar under consideration is loaded only statically.*

1. Introduction

Loss of elastic stability in linear continuum systems is in general a dynamical proces even if the structure is loaded only statically. The most safe method to asses the stability includes the application of dynamic criterion of stability in which inertia effects are included. However, there is a broad class of important structural systems including conservative systems which can be succesfully solved with the aid of substantially simpler static criterion.

The applicability of the static criterion is closely connected to the selfadjointness of the corresponding boundary value problem. For a broad class of boundary problems including eigenvalue problems of the type $Lw + \lambda w = 0$ the selfadjointness together with positive definitivenes is the sufficient condition to apply the static criterion, as the complete spectrum is real valued. In this paper we classify some of non-trivial boundary conditions for coupled bending-torsional vibration of a bar in compression and torsion according to their selfadjointness or nonselfadjointness in order to asses the applicability of the static criterion in advance.

2. Governing equations of a bar in compression and torsion

Let us consider a straight slender elastic prismatic bar loaded at unconstrained end by a compressive force **P** of constant magnitude *P* and by a torque **M** of constant magnitude *M*, Figure 1. The directional aspects of the behaviour of vectors **P** and **M** we specify later in boundary conditions. In sufficiently slender bar considerable bending deformations may appear as a consequence of external loading **P** and/or **M** with levels exceeding certain critical

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values. The originally straight centerline of of a bar in equilibrium may be transformed into spatial curve. Let the *x*-axis of the coordinate system is aligned along the undeformed centerline, the *y* and *z*-axis are aligned with principal inertia axes of the cross-section. For simplicity constant circular cross-section is considered to have equal flexural rigidity *EJ* in both bending planes. Then the governing equations have constant coefficients and for harmonic regime are of form

$$\begin{aligned} y^{(4)}(x)+\alpha y^{(2)}(x)+\mu z^{(3)}(x) &= \lambda^2 y(x) \\ z^{(4)}(x)+\alpha z^{(2)}(x)-\mu z^{(3)}(x) &= \lambda^2 z(x), \end{aligned} \tag{1}$$

where *y(x)* and *z(x)* are the flexural deflections in planes *xy* and *xz*, respectively. $\mu = Ml/EJ$ is the nondimensional torque, $\alpha = Pl^2/EJ$ is the constant nondimensional compressive force and $\lambda^2 = \omega^2 m/EJ$ is the nondimensional frequency parameter with ω standing for the circular frequency. Detailed derivation of the governing equation can be found in Bolotin (1961).

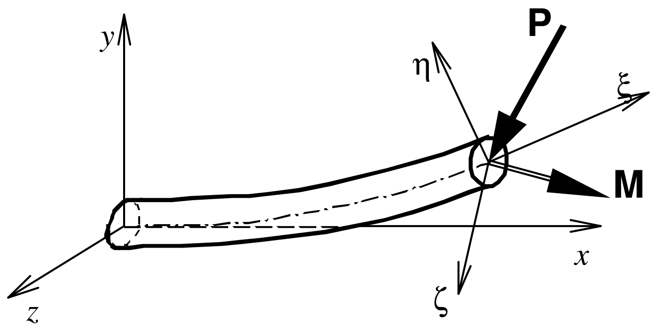

Figure 1. The absolute and the local coordinate system.

For two point boundary problems the boundary conditions at each end $s = 0,1$ are prescribed in general as a set of four linear forms at each end

$$U_{is}[y(s), y^{(1)}(s), y^{(2)}(s), y^{(3)}(s), z(s), z^{(1)}(s), z^{(2)}(s), z^{(3)}(s)] = \sum_{j=0}^{3}\left(a_{ijs}y^{(j)}(s)+b_{ijs}z^{(j)}(s)\right)=0, \tag{2}$$

$$\sum_{j=0}^{3}\left(\left|a_{ijs}\right|+\left|b_{ijs}\right|\right)\neq 0$$

where the coeficients a_{ijs}, b_{ijs} of the linear forms U_{is} are usually constants, however, in practical applications they can be functions of any design parameters like α, μ, frequency or any other parameter.

3. Analysis of the boundary eigenvalue problem

Here we concentrate on the selfadjointness of the given boundary value problem. Evidently the linear differential expression representating the left hand of the governing equation (1) is symmetrical in even derivations according to the spatial coordinate *x* and antisymmetrical in odd derivations, so the differential expression *L* is in the definition domain *(0,1)* formally selfadjoint. Then the selfadjointness of the boundary eigenvalue problem as a whole is decided only by the boundary conditions, what results in decision on the symmetry or nonsymmetry of the Dirichlets remainder term. This term is obtained from trivial integration by parts of the expression *(LW,U)* and subsequent transformations with respect to prescribed four linearly independent boundary conditions

$$U_{i_1}=0,\quad U_{i_2}=0,\quad U_{i_3}=0,\quad U_{i_4}=0,\qquad 1\le i_1\le i_2\le i_3\le i_4\le 8. \tag{3}$$

Then from the rearranged Dirichlets remainder

$$M(W,U)\big|_s=\overline{W}_{1s}U_{8s}+\overline{W}_{2s}U_{7s}+\overline{W}_{3s}U_{6s}+\overline{W}_{4s}U_{5s}+\overline{W}_{5s}U_{4s}+\overline{W}_{6s}U_{3s}+\overline{W}_{7s}U_{2s}+\overline{W}_{8s}U_{1s}=0$$

obviously the adjoint boundary conditions are expressed by four linear forms (4)

$$W_{j1s}=0,\quad W_{j2s}=0,\quad W_{j3s}=0,\quad W_{j4s}=0,\qquad jks=9-iks \tag{5}$$

so that the total of eight conditions (3,5) guarantee the vanishing of the Dirichlets remainder term (4). If the systems of linear forms U_{is} and W_{is} are equivalent, then the corresponding boundary value problem is selfadjoint.

4. Conditions of selfadjointness of dynamical boundary conditions

Dynamical boundary conditions specify in detail the behaviour of the loading force **P** and of the loading torque **M** throughout the process of the deformation of the bar. Formally they express the relations between the internal forces and torques

$$T_\xi=-\alpha, T_\eta=-y^{(3)}-\mu z^{(2)}, T_\zeta=-z^{(3)}+\mu y^{(2)}, S_\xi=\mu, S_\eta=-z^{(2)}, S_\zeta=y^{(2)} \tag{6}$$

on one side and the possible combinations of geometrical variables *y(s),* $y^{(1)}$ *(s), z(s)* and $z^{(1)}$(s) on the other side. For the sign convention of internal forces see Figure 2.

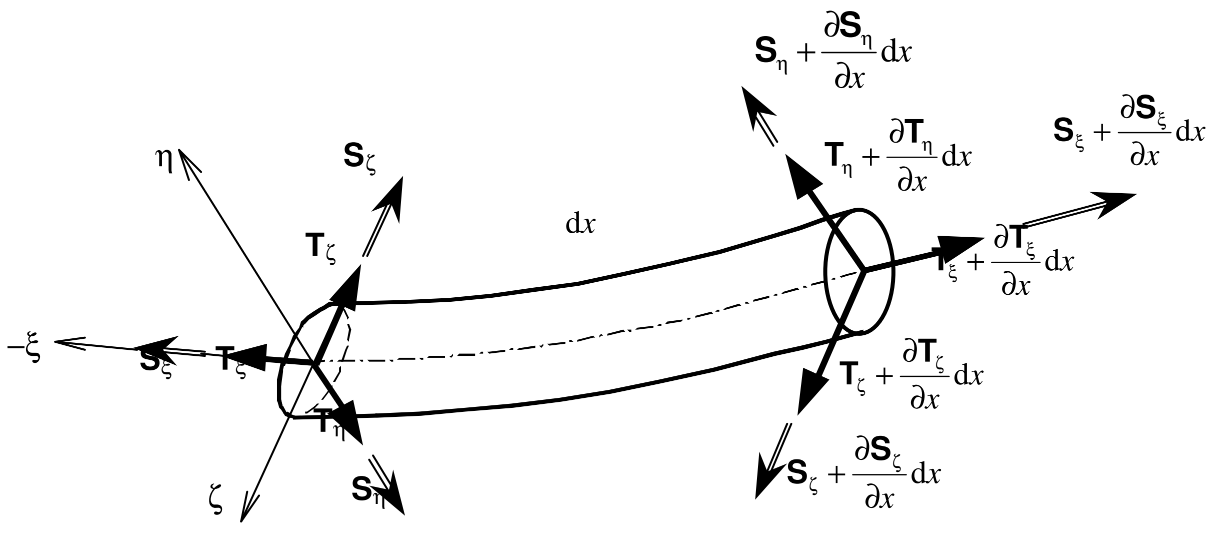

Figure 2. Sign convention for internal forces and moments.

Sufficiently general class of dynamical boundary conditions is given by relation

$$\begin{bmatrix} y^{(3)}(s)+\mu z^{(2)} \\ y^{(2)}(s) \\ z^{(3)}(s)-\mu y^{(2)}(s) \\ z^{(2)}(s) \end{bmatrix} + \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \begin{bmatrix} y(s) \\ y^{(1)}(s) \\ z(s) \\ z^{(1)}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

in which t_{ij} are suitable coefficients characterizing the particular boundary condition. Substituting to the transformed Dirichlets remainder leads after minor manipulations to adjoint boundary conditions

$$\begin{bmatrix} u^{(3)}(s)+\mu v^{(2)} \\ u^{(2)}(s) \\ v^{(3)}(s)-\mu u^{(2)}(s) \\ v^{(2)}(s) \end{bmatrix} + \begin{bmatrix} t_{11} & \alpha - t_{21} & t_{31} & -t_{41} \\ \alpha - t_{12} & t_{22} & -t_{32} & \mu + t_{42} \\ t_{13} & -t_{23} & t_{33} & \alpha - t_{43} \\ -t_{14} & -\mu + t_{24} & \alpha - t_{34} & t_{44} \end{bmatrix} \begin{bmatrix} u(s) \\ u^{(1)}(s) \\ v(s) \\ v^{(1)}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

As both the given as well as the adjoint boundary conditions are of canonical form, the follo-wing conditions are sufficient and necessary conditions of the selfadjointness of the dyna-mical boundary conditions (7)

$$t_{12} = \alpha - t_{21},\ t_{24} = \mu + t_{42},\ t_{34} = \alpha - t_{43},\ t_{13} = t_{31},\ t_{14} = -t_{41},\ t_{23} = -t_{32} \tag{9}$$

Note, that there are no constraints on diagonal coefficients $t_{11}, t_{22}, t_{33}, t_{44}$. Fo details on matrix operations on Dirichlets remainder see Nánási (1994).

To study the influence of the behaviour of loading force **P** and torque **M** at the end of the bar on the conservative or nonconservative nature of boundary eigenvalue problem we restrict the general form to the following simpler relations

$$\begin{aligned} y^{(3)}(s) + \mu z^{(2)}(s) + \theta_y \alpha y^{(1)}(s) &= 0 \\ y^{(2)}(s) + \psi_y \mu z^{(1)}(s) + \varepsilon_y \alpha y(s) &= 0 \\ z^{(3)}(s) - \mu y^{(2)}(s) + \theta_z \alpha z^{(1)}(s) &= 0 \\ z^{(2)}(s) - \psi_z \mu y^{(1)}(s) + \varepsilon_z \alpha z(s) &= 0 \end{aligned} \tag{10}$$

in which the parameters $\theta_y, \theta_z, \varepsilon_y, \varepsilon_z, \psi_y, \psi_z$ characterize the behaviour of the loading vectors **P**, **M** with respect to deflected center line.

Adjoint boundary conditions corresponding to prescribed conditions (10) are of form

$$\begin{aligned} u^{(3)}(s) + \mu v^{(2)}(s) + (1-\varepsilon_y)\alpha u^{(1)}(s) &= 0 \\ u^{(2)}(s) + (1-\psi_z)\mu v^{(1)}(s) + (1-\theta_y)\alpha u(s) &= 0 \\ v^{(3)}(s) - \mu u^{(2)}(s) + (1-\varepsilon_z)\alpha v^{(1)}(s) &= 0 \\ v^{(2)}(s) - (1-\psi_y)\mu u^{(1)}(s) + (1-\theta_z)\alpha v(s) &= 0 \end{aligned} \tag{11}$$

The conditions for the selfadjointness (x) are now reduced to the set of three relations

$$\theta_y + \varepsilon_y = 1, \quad \theta_z + \varepsilon_z = 1, \quad \psi_y + \psi_z = 1. \tag{12}$$

The most general case of seladjoint boundary conditions of the class (10) has the form

$$
\begin{aligned}
&y^{(3)}(s)+\mu z^{(2)}(s)+\theta_y \alpha y^{(1)}(s)=0\\
&y^{(2)}(s)+\psi_y \mu z^{(1)}(s)+\left(1-\theta_y\right)\alpha y(s)=0\\
&z^{(3)}(s)-\mu y^{(2)}(s)+\theta_z \alpha z^{(1)}(s)=0\\
&z^{(2)}(s)-\left(1-\psi_y\right)\mu y^{(1)}(s)+\left(1-\theta_z\right)\alpha z(s)=0
\end{aligned}
\tag{13}
$$

5. Interpretation of the possible boundary conditions

Using expressions (6), the seladjoint boundary conditions can be written in terms of internal forces:

$$
\begin{aligned}
&T_\eta=\theta_y \alpha y^{(1)}, \quad S_\zeta=-\psi_y \mu z^{(1)}-\left(1-\theta_y\right)\alpha y,\\
&T_\zeta=\theta_z \alpha z^{(1)}, \quad S_\eta=-\left(1-\psi_y\right)\mu y^{(1)}+\left(1-\theta_z\right)\alpha z
\end{aligned}
\tag{14}
$$

To understand their physical meaning it is useful to rewrite them also in their original form

$$
\begin{aligned}
&T_\eta=\theta_y \alpha y^{(1)}, \quad S_\zeta=-\psi_y \mu z^{(1)}-\varepsilon_y \alpha y,\\
&T_\zeta=\theta_z \alpha z^{(1)}, \quad S_\eta=-\psi_z \mu y^{(1)}+\varepsilon_z \alpha z
\end{aligned}
\tag{15}
$$

Parameters θ_y, θ_z describe the orientation of the compressive force **P** with respect to the absolute or local coordinate system, see Figure 1. The role of parameters ψ_y, ψ_z is similar, they give information on spatial orientation of the torque **M**, while parameters ε_y, ε_z are the nondimensional measure of the excentricity of the compressive force **P**.

From the above relations it follows, that for selfadjointness the component $\varepsilon_y y$ of the eventual excentricity of the compressive force is coupled with the component $\theta_y y^{(1)}$ of inclination of the compressive force from the centerline, the same holds for components $\varepsilon_z z$ and $\theta_z z^{(1)}$. For a torque **M** to act in a conservative manner the condition $\psi_y + \psi_z=1$ requires specific coupling between inclinations $\psi_z y^{(1)}$ and $\psi_y z^{(1)}$ which is difficult to maintain mechanically without electronic control system.

Easier to interprete are special cases of the class (10) obtained when for parameters θ_y, θ_z, ε_y, ε_z, ψ_y, ψ_z the values *0* or *1* are chosen.

$\theta_y = 1$ and/or $\theta_z = 1$

Under the boundary condition $T_\eta = \alpha y^{(1)}$ the projection $P_{\xi\eta}$ of the force **P** on plane $\xi\eta$ is in every instant declined from the local direction ξ by the angle $y^{(1)}$, as depicted in Figure 3a. Similar is the interpretation of the case $\theta_z = 1$ when $T_\zeta = \alpha z^{(1)}$, now the projection $P_{\xi\zeta}$ of the force **P** on plane $\xi\zeta$ is declined from the local direction ξ by the angle $z^{(1)}$, Figure 3b. If both conditions $\theta_y = 1$ and $\theta_z = 1$ hold, then the force **P** has constant direction aligned with the *x*-direction of the absolute coordinate system. This is a typical conservative force and for $\mu = 0$ the eigenvalue problem is selfadjoint, therefore the Euler method (static criterion) can be used. For coupled problem $\mu \neq 0$ the selfadjointness requires to orient the torque **M** according to the condition $\psi_y + \psi_z=1$.

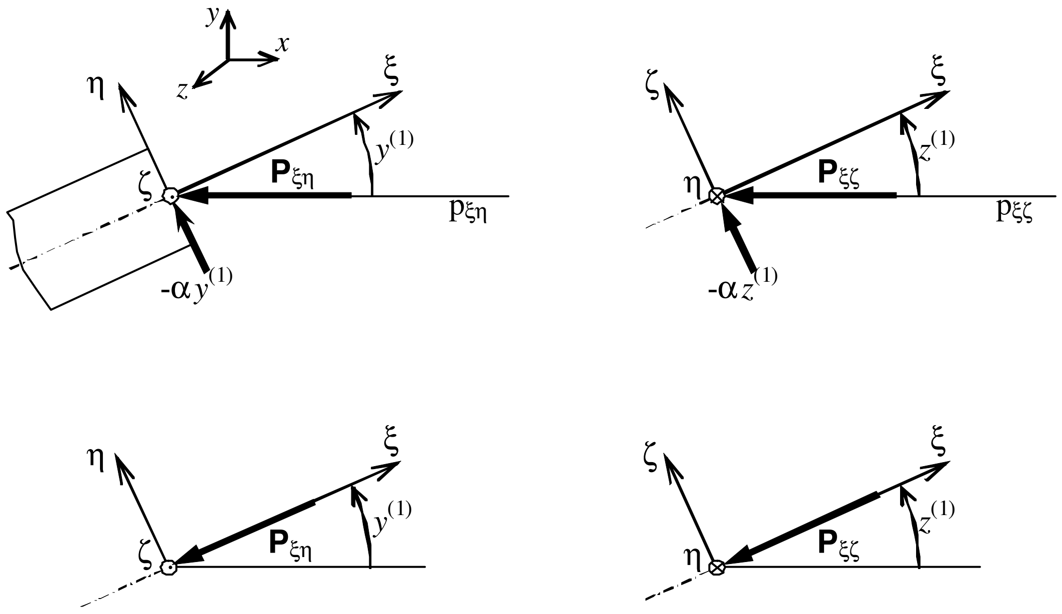

Figure 3. Behaviour of the compressive force

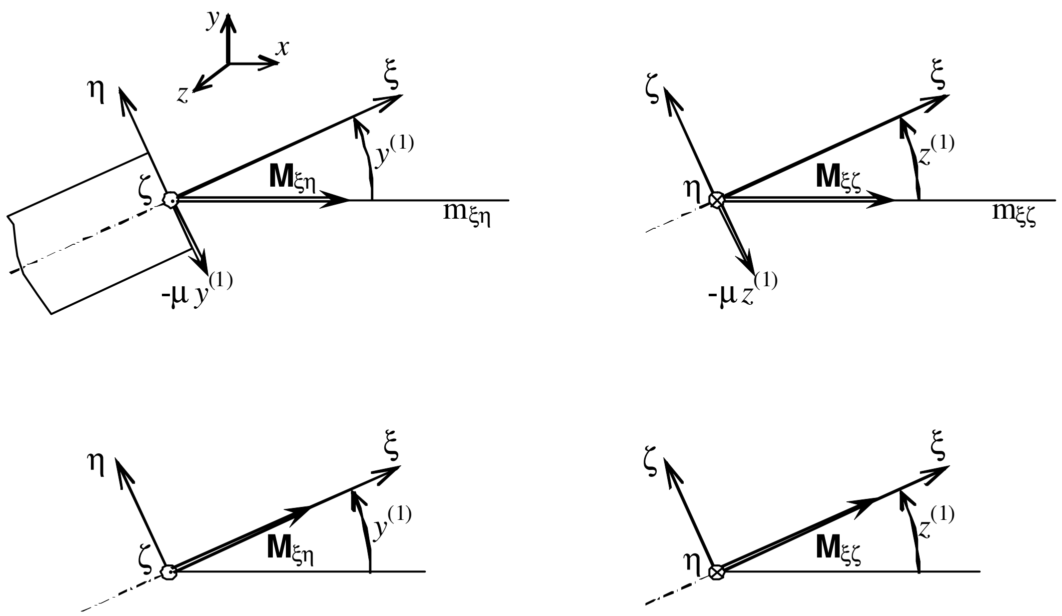

Figure 4. Behaviour of the loading torque

$\theta_y = 0$ and/or $\theta_z = 0$

With $\theta_y = 0$ we have $T_\eta = 0$, now the projection $P_{\xi\eta}$ of the force **P** on plane $\xi\eta$ is in every instant orthogonal to the direction η in the local coordinate system, Figure 3c. Interpretation of the case $\theta_z = 0$ with $T_\zeta = 0$ is similar, see Figure 3d. When both conditions $\theta_y = 0$ and $\theta_z = 0$ hold, then the force **P** is a typical follower force, her direction is aligned with the direction of the deformed centerline. This is a typical nonconservative problem, for $\mu = 0$ is known as

Beck`s beam. The eigenvalue problem is nonselfadjoint regardless of the behaviour of the torque **M**.

$\theta_y = 1$, $\theta_z = 0$* or *$\theta_y = 0$, $\theta_z = 1$

Under conditions $\theta_y = 1$, $\theta_z = 0$ the vector **P** is at every instant confined to the plane $\xi\eta$ of the local coordinate system and in this plane oscillates around the local axis ξ. For $\theta_y = 0$ and $\theta_z = 1$ we have similar confinement to the plane $\xi\zeta$ of the local coordinate system. Both cases are in general nonselfadjoint, unless suitable excentricity of the point of application of the compressive force **P** is provided.

$\psi_y = 1$, $\varepsilon_y = 0$* or *$\psi_z = 1$, $\varepsilon_z = 0$

For $\psi_y = 1$, $\varepsilon_y = 0$ we have $S_\zeta = -\mu z^{(1)}$, then the projection $M_{\xi\eta}$ of the torque **M** to the plane $\xi\eta$ is declined from the local ξ- axis by the angle $z^{(1)}$, Figure 4b. Similar case $\psi_z = 1$, $\varepsilon_z = 0$ is on Figure 4a. With both $\psi_y = 1$ and $\psi_z = 1$ the torque **M** has constant direction with respect to the absolute coordinate system regardless of the deflections, however, now the boundary conditions are nonselfadjoint unlike the case of similar behaviour of the force **P**.

$\psi_y = 0$, $\varepsilon_y = 0$* or *$\psi_z = 0$, $\varepsilon_z = 0$

With $\psi_y = 0$, $\varepsilon_y = 0$ we have $S_\zeta = 0$, now the projection $M_{\xi\eta}$ of the torque **M** to the plane $\xi\eta$ is orthogonal to the local η- axis, Figure 4d. Similar case $\psi_z = 0$, $\varepsilon_z = 0$ is on Figure 4c. With both $\psi_y = 0$ and $\psi_z = 0$ the torque **M** has constant direction with respect to the local coordinate system regardless of the deflection – the torque follows the deflected centerline. Boundary conditions are nonselfadjoint. With $\alpha = 0$ the coupled problem reduces to the Greenhill`s problem, which is nonconservative.

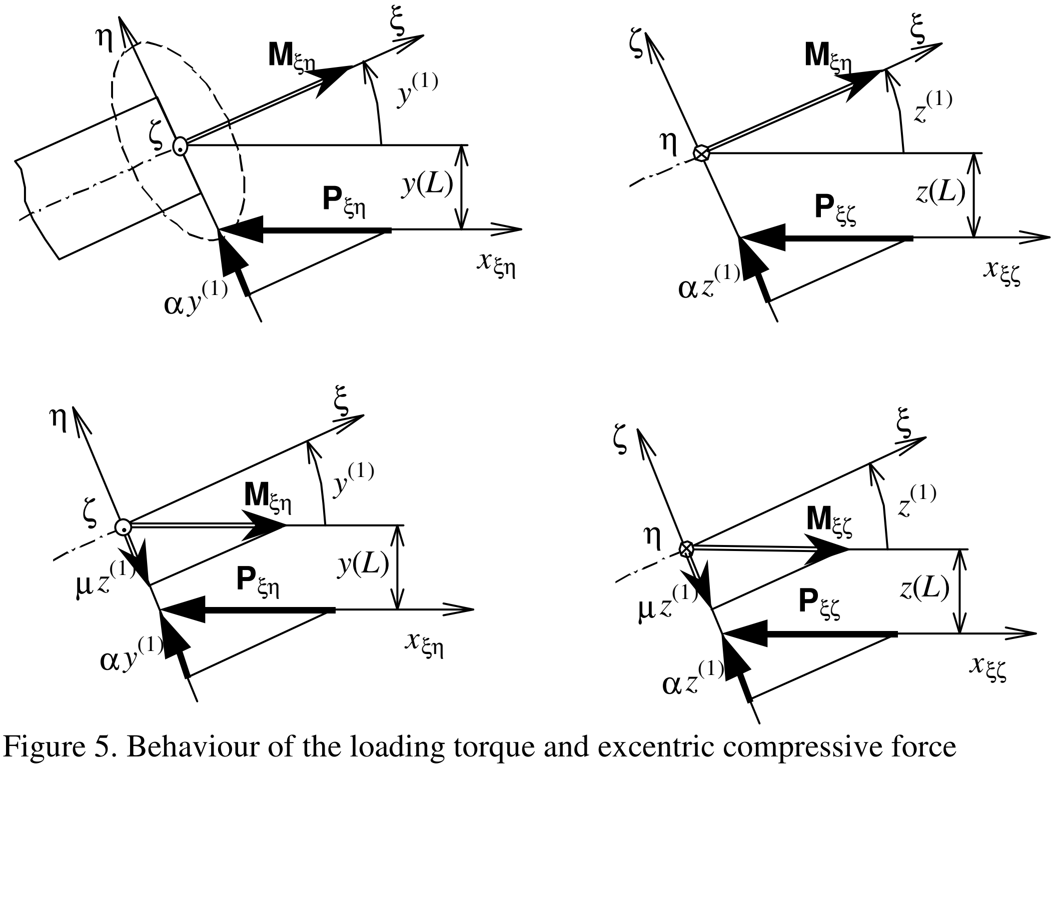

Figure 5. Behaviour of the loading torque and excentric compressive force

$\theta_y = 1,\ \psi_y = 0,\ \varepsilon_y = 1$

In this case we have boundary conditions $T_\eta = \alpha\, y^{(1)}$ and $S_\zeta = -\alpha y$. The possible realization of such boundary conditions is sketched on Figure 5a. The torque has the nature of follower load. To ensure the excentricity y, a platform fixed to the end of the beam is required. This case corresponds to the nonconservative Reut`s problem, when $\mu = 0$ and the motion is restricted on the plane xy. Similar case $\theta_z = 1,\ \psi_z = 0,\ \varepsilon_z = 1$ is explained by Figure 5b.

$\theta_y = 1,\ \psi_y = 1,\ \varepsilon_y = 1$

This case differs from the previous one only by the behaviour of the torque, which now has constant direction with respect to the absolute coordinate system, see Figure 5c,d. Neverthereless, this boundary conditions are nonselfadjoint.

6. Conclusion

For vibration of coupled bending-torsional vibration of a bar in compression and torsion general dynamic boundary conditions have been formulated to investigate the influence of the behaviour of the compressive force and torque on selfadjointness of the corresponding eigenvalue problem. Adjoint boundary conditions have been derived and conditions of selfadjointness have been deduced. The general form of selfadjoint boundary conditions depends on three independent parameters describing the eventual excentricity of the compressive force and angular aspects of the behaviour of both the loading force and torque. As the general selfadjoint boundary conditions are difficult to interprete physically as well as to realize mechanically, a series of simplified cases have been analysed. We conclude, that most natural formulations lead to nonseladjoint boundary conditions and the corresponding problems are nonconservative. Most of selfadjoint cases require specific behaviour of the torque, which can be achieved only in artificial way.

7. Acknowledgement

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8. References

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