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Modelling of Imperfect Bond Between Composite Matrix and Reinforcement by the FETI Method

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Summary:*Composite materials consist of a composite matrix and a reinforcement usually created by fibres. The overall properties of a composite material strongly depend on interaction between the composite matrix and the reinforcement. The contribution describes the interaction by so-called dual formulation where the original unknowns, in mechanical case displacements, are replaced by the dual unknowns, in mechanical case forces. The dual formulation is more robust and the conditions on interfaces are simpler than conditions in the original formulation. The approach is based on the FETI method which was developed as a domain decomposition method in 1991 by C. Farhat and co-workers.*

1. Introduction

The contribution deals with modelling of interaction of the composite matrix and reinforcement which are basic constituents of a composite material. The notion of a composite material describes not only the classical composites but also materials such as reinforced or prestressed concrete. The composite materials are used in many areas of mechanical engineering, aircraft engineering, civil engineering, etc. The composite materials are studied carefully and the modelling of interaction between reinforcement and the composite matrix is an inseparable part of composite analysis.

The simplest analyses assume the perfect bond between the composite matrix and reinforcement. There are problems, where the perfect bond is adequate description of reality. Even the perfect bond is the simplest case, it can cause numerical difficulties when an inappropriate numerical model is used. Imperfect bond describes the reality better but it leads to more serious numerical difficulties than the perfect bond. The choice of appropriate numerical model is more important in this case.

The modelling of the interaction is based on pullout tests. The arrangement of such tests is the following. There is a composite matrix with one embedded fibre which is under tension.

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The growing force in the fibre causes debonding of matrix-fibre connection and fibre moves out from the matrix. Detailed description of pullout effects is relatively complicated and several simplified approaches are used. This contribution deals with the case with perfect bonding between reinforcement and matrix as well as debonding which is controlled by a linear relationship. The most general model with nonlinear debonding is not studied, but it is in the centre of our attention.

2. Overview of the FETI method

The FETI method was introduced by Farhat and Roux in 1991 in Farhat and Roux [1991]. It is a non-overlapping domain decomposition method which enforces the continuity among subdomains by Lagrange multipliers. The FETI method or its variants have been applied to broad class of two and three dimensional problems of second and fourth order. More details can be found e.g. in Toselli and Widlund [2005], Farhat and Roux [1994], Kruis [2006], Rixen et al. [1999], Bhardwaj et al. [2000].

Let the original domain be decomposed to m subdomains. Unknown displacements defined on the j-th subdomain are located in the vector $\boldsymbol{u}^j$. All unknown displacements are located in the vector

$$\boldsymbol{u}^T = \left((\boldsymbol{u}^1)^T, (\boldsymbol{u}^2)^T, \ldots, (\boldsymbol{u}^m)^T\right) \tag{1}$$

The stiffness matrix of the j-th subdomain is denoted $\boldsymbol{K}^j$ and the stiffness matrix of the whole problem has the form

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}^1 & & & \\ & \boldsymbol{K}^2 & & \\ & & \ddots & \\ & & & \boldsymbol{K}^m \end{pmatrix} \tag{2}$$

The nodal loads of the j-th subdomain are located in the vector $\boldsymbol{f}^j$ and the load vector of the problem has the form

$$\boldsymbol{f}^T = \left((\boldsymbol{f}^1)^T, (\boldsymbol{f}^2)^T, \ldots, (\boldsymbol{f}^m)^T\right) \tag{3}$$

Continuity among subdomains has the form

$$\boldsymbol{B}\boldsymbol{u} = \boldsymbol{0} \tag{4}$$

where the boolean matrix $\boldsymbol{B}$ has the form

$$\boldsymbol{B} = \left(\boldsymbol{B}^1, \boldsymbol{B}^2, \ldots, \boldsymbol{B}^m\right) \tag{5}$$

The matrices $\boldsymbol{B}^j$ contain only entries equal to $1, -1, 0$. With the previously defined notation, the energy functional has the form

$$\Pi(\boldsymbol{u}, \boldsymbol{\lambda}) = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{f} + \boldsymbol{\lambda}^T\boldsymbol{B}\boldsymbol{u} \tag{6}$$

where the vector $\boldsymbol{\lambda}$ contains Lagrange multipliers. Stationary conditions of the energy functional have the form

$$\frac{\partial \Pi}{\partial \boldsymbol{u}} = \boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} + \boldsymbol{B}^T\boldsymbol{\lambda} = \boldsymbol{0} \tag{7}$$

$$\frac{\partial \Pi}{\partial \boldsymbol{\lambda}} = \boldsymbol{B}\boldsymbol{u} = \boldsymbol{0} \tag{8}$$

Equation (7) expresses the equilibrium condition while (8) expresses the continuity condition. The known feature of the FETI method is application of a pseudoinverse matrix in relationship for unknown displacements

$$\boldsymbol{u} = \boldsymbol{K}^+\left(\boldsymbol{f} - \boldsymbol{B}^T\boldsymbol{\lambda}\right) + \boldsymbol{R}\boldsymbol{\alpha} \tag{9}$$

which stems from floating subdomains. The stiffness matrix of a floating subdomain is singular. The matrix $\boldsymbol{R}$ contains the rigid body modes of particular subdomains and the vector $\boldsymbol{\alpha}$ contains amplitudes that specifies the contribution of the rigid body motions to the displacements. The pseudoinverse matrix and the rigid body motion matrix can be written in the form

$$\boldsymbol{K}^+ = \begin{pmatrix} (\boldsymbol{K}^1)^+ & & & \\ & (\boldsymbol{K}^2)^+ & & \\ & & \ddots & \\ & & & (\boldsymbol{K}^m)^+ \end{pmatrix} \quad \boldsymbol{R} = \begin{pmatrix} \boldsymbol{R}^1 & & & \\ & \boldsymbol{R}^2 & & \\ & & \ddots & \\ & & & \boldsymbol{R}^m \end{pmatrix} \tag{10}$$

Except of utilisation of the pseudoinverse matrix, a solvability condition in the form

$$\left(\boldsymbol{f} - \boldsymbol{B}^T\boldsymbol{\lambda}\right) \perp \ker \boldsymbol{K} = \boldsymbol{R} \tag{11}$$

has to be taken into account. Substitution of unknown displacements to the continuity condition leads to the form

$$\boldsymbol{B}\boldsymbol{K}^+\boldsymbol{B}^T\boldsymbol{\lambda} = \boldsymbol{B}\boldsymbol{K}^+\boldsymbol{f} + \boldsymbol{B}\boldsymbol{R}\boldsymbol{\alpha} \tag{12}$$

The solvability condition can be written in the form

$$\boldsymbol{R}^T\left(\boldsymbol{f} - \boldsymbol{B}^T\boldsymbol{\lambda}\right) = \boldsymbol{0} \tag{13}$$

Usual notation in the FETI method is the following

$$\boldsymbol{F} = \boldsymbol{B}\boldsymbol{K}^+\boldsymbol{B}^T \tag{14}$$

$$\boldsymbol{G} = -\boldsymbol{B}\boldsymbol{R} \tag{15}$$

$$\boldsymbol{d} = \boldsymbol{B}\boldsymbol{K}^+\boldsymbol{f} \tag{16}$$

$$\boldsymbol{e} = -\boldsymbol{R}^T\boldsymbol{f} \tag{17}$$

The continuity and solvability conditions can be rewritten with the defined notation in the form

$$\begin{pmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \boldsymbol{d} \\ \boldsymbol{e} \end{pmatrix} \tag{18}$$

The system of equations (18) is called the coarse or interface problem.

3. Modification of the method

The classical FETI method uses the continuity condition (4) which enforces the same displacements at the boundary nodes. If there is a reason for different displacements between two neighbour subdomains, the continuity condition transforms itself to a slip condition. The slip condition can be written in the form

$$\boldsymbol{B}\boldsymbol{u} = \boldsymbol{s} \tag{19}$$

The vector $\boldsymbol{s}$ stores slips between boundary nodes. For this moment, the slip is assumed to be prescribed and constant.

Let the boundary unknowns be split to two disjunct parts. The boundary unknowns which satisfy the continuity condition are located in the vector $\boldsymbol{u}_c$, while the boundary unknowns which satisfy the slip condition are located in the vector $\boldsymbol{u}_s$. Similarly to the continuity condition in the FETI method, the vectors $\boldsymbol{u}_c$ and $\boldsymbol{u}_s$ can be written in the form

$$\boldsymbol{u}_c = \boldsymbol{B}_c\boldsymbol{u} \tag{20}$$

$$\boldsymbol{u}_s = \boldsymbol{B}_s\boldsymbol{u} \tag{21}$$

where $\boldsymbol{B}_c$ and $\boldsymbol{B}_s$ are the boolean matrices. Now, the continuity condition has the form

$$\boldsymbol{B}_c\boldsymbol{u} = \boldsymbol{0} \tag{22}$$

and the slip condition has the form

$$\boldsymbol{B}_s\boldsymbol{u} = \boldsymbol{s} \tag{23}$$

The conditions (22) and (23) can be amalgamated to a new interface condition

$$\boldsymbol{B}\boldsymbol{u} = \begin{pmatrix} \boldsymbol{B}_c \\ \boldsymbol{B}_s \end{pmatrix} \boldsymbol{u} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s} \end{pmatrix} = \boldsymbol{c} \tag{24}$$

The energy functional can be rewritten to the form

$$\Pi = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{u}^T\boldsymbol{f} + \boldsymbol{\lambda}^T(\boldsymbol{B}\boldsymbol{u} - \boldsymbol{c}) \tag{25}$$

The stationary conditions have the form

$$\boldsymbol{K}\boldsymbol{u} - \boldsymbol{f} + \boldsymbol{B}^T\boldsymbol{\lambda} = \boldsymbol{0} \tag{26}$$

$$\boldsymbol{B}\boldsymbol{u} = \boldsymbol{c} \tag{27}$$

As was mentioned before, the system of two stationary conditions is accompanied by the solvability condition (11). The expression of the vector $\boldsymbol{u}$ given in (9) remains the same and the interface conditions has the form

$$\boldsymbol{B}\boldsymbol{K}^+\boldsymbol{B}^T\boldsymbol{\lambda} = \boldsymbol{B}\boldsymbol{K}^+\boldsymbol{f} + \boldsymbol{B}\boldsymbol{R}\boldsymbol{\alpha} - \boldsymbol{c} \tag{28}$$

and the solvability condition has the form

$$\boldsymbol{R}^T\left(\boldsymbol{f} - \boldsymbol{B}^T\boldsymbol{\lambda}\right) = \boldsymbol{0} \tag{29}$$

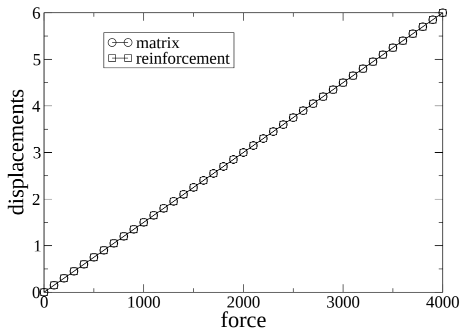

Figure 1: Perfect bond.

The coarse problem can be written with the help of notation (14) - (17) in the form

$$\begin{pmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \boldsymbol{d} - \boldsymbol{c} \\ \boldsymbol{e} \end{pmatrix} \tag{30}$$

The modified coarse problem (30) differs from the original coarse problem (18) by the vector of prescribed slips $\boldsymbol{c}$ on the right hand side.

The prescribed slip between two subdomains is not a common case. On the other hand, the slip often depends on shear stress. Discretized form of equations used in the coarse problem requires a discretized law between slip as a difference of two neighbour displacements and nodal forces as integrals of stresses along element edges. One of the simplest law is the linear relationship

$$\boldsymbol{c} = \boldsymbol{H}\boldsymbol{\lambda} \tag{31}$$

where $\boldsymbol{H}$ denotes the compliance matrix. Substitution of (31) to the coarse problem (30) leads to the form

$$\begin{pmatrix} \boldsymbol{F} + \boldsymbol{H} & \boldsymbol{G} \\ \boldsymbol{G}^T & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{pmatrix} = \begin{pmatrix} \boldsymbol{d} \\ \boldsymbol{e} \end{pmatrix} \tag{32}$$

It should be noted that the coarse system of equations (32) is usually solved by the modified conjugate gradient method. Details can be found in Farhat and Roux [1994] and Rixen et al. [1999]. The only difference with respect to the system (18) is the compliance matrix $\boldsymbol{H}$. Only one step, the matrix-vector multiplication, of the modified conjugate gradient method should be changed. The compliance matrix may be a diagonal or nearly diagonal matrix.

4. Numerical examples

Four cases of bonding/debonding behaviour are computed by the classical and modified FETI method. There are always two subdomains. One subdomain represents the composite matrix and the second one represents the fibre. A perfect bond is described directly by the classical FETI method. The usual continuity condition is used. The displacements of the fibre and composite matrix at selected point are identical and the situation is depicted in Figure 1.

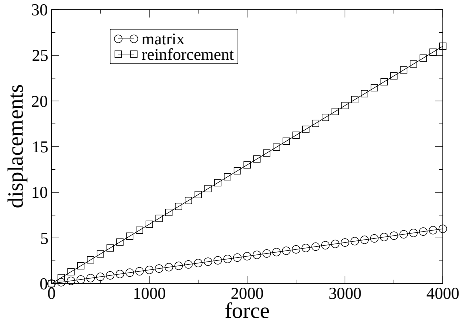

Figure 2: Imperfect bond (debonding).

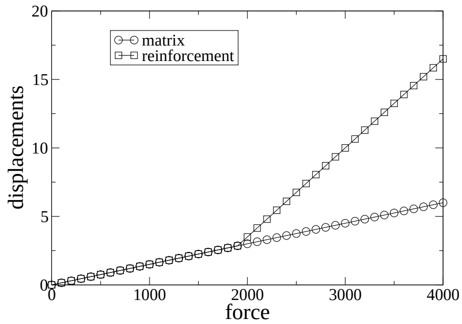

Figure 3: Imperfect bond with delay.

An imperfect bond is described by the modified FETI method with the constant compliance matrix $\boldsymbol{H}$. The displacements of fibre are greater than the displacements of composite matrix. The greater force is applied, the greater slip occurs. The situation is depicted in Figure 2.

A perfect bond followed by an imperfect bond is modelled by the modified FETI method. At the beginning, the compliance matrix is zero matrix which expresses infinitely large stiffness between subdomains. At a certain load level, debonding effect is assumed and the compliance matrix is redefined and it is a constant matrix in the following steps. The displacements of the fibre and matrix are the same at the beginning but then they are different. The situation is depicted in Figure 3.

The last example shows similar problem as the previous one. The compliance matrix $\boldsymbol{H}$ is not assumed constant but the compliances are growing from zero values up to a certain level. It means, that the stiffness is decreasing from infinitely large value to some finite value. The greater force acts, the higher compliance is attained and greater slip between the fibre and composite matrix occurs. The situation is depicted in Figure 4.

5. Conclusions

A slight modification of the FETI method is proposed for problems with the imperfect bond between the composite matrix and reinforcement. The perfect bond is modelled by the classical FETI method. Application of a constant compliance matrix leads to linear debonding while

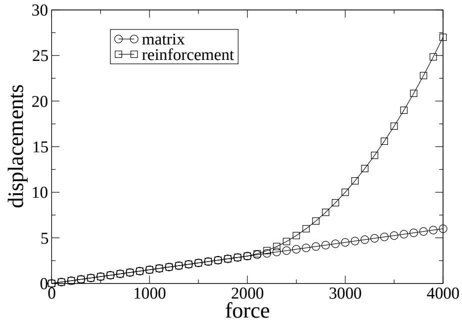

Figure 4: Imperfect bond with changing compliance.

a variable compliance matrix can describe nonlinear debonding effects. The advantage of the proposed modification stems from the structure of the compliance matrix which can be nearly diagonal and therefore computationally cheap. The second advantage stems from possible parallelisation. Each fibre, generally each piece of reinforcement, as well as the composite matrix can be assigned to one processor and large problems may be solved efficiently.

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