National Conference with International Participation
ENGINEERING MECHANICS 2007
Svratka, Czech Republic, May 14 – 17, 2007

ENERGY CONJUGATE MEASURES OF STRESS AND STRAIN IN CONTINUUM MECHANICS

Z. Hrubý[1], J. Plešek[2]

Summary: *The energy conjugate measures of stress and strain enter any constitutive relation for the material behaviour description. There are many strain measures in common use, several of them discussed in the text. The main aim of the contribution is to focus on the derivation of a general energy conjugate stress tensor corresponding with a general Lagrangean strain tensor.*

1. Introduction

The Cauchy stress tensor and the symmetric part of the velocity gradient enter the formula for the stress power. This takes place in the area of so-called current configuration. Whenever a problem of non-linear geometry is formulated in continuum mechanics, either the total or the updated Lagrangean formulation is used, therefore, the stress power is expressed in the referential configuration. Stress and strain measures are considered to be energy conjugate if obeying equality (Khan & Huang, 1995)

$$\int_V \boldsymbol{\sigma} : \mathbf{D}\, \mathrm{d}V = \int_{V_0} \boldsymbol{\Sigma} : \dot{\mathbf{E}}\, \mathrm{d}V_0 \,, \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{D}$ the symmetric part of velocity gradient, $\mathbf{E}$ a general Lagrangean strain tensor, $\boldsymbol{\Sigma}$ a general energy conjugate stress tensor with respect to $\mathbf{E}$, V the volume in the current configuration, and V_0 the volume in the referential configuration. The time rate on the right side of equation (1) is essential. It can be proved that for instance the second Piola-Kirchhoff stress and the Green-Lagrange strain tensor are energy conjugate as well as the first Piola-Kirchhoff stress tensor and the deformation gradient.

In case of small deformation, neither the total nor the updated Lagrangean formulation is necessary to be performed. The theory of linear continuum is used in these cases and it is assumed there is a negligible difference between the current and the referential configuration. Only in such a case the Cauchy stress tensor and the infinitesimal strain tensor are approximately considered to be conjugate, although there is definitely no energy conjugate strain tensor corresponding with the Cauchy stress tensor.

---

[1] Ing. Zbyněk Hrubý, Institute of Thermomechanics AS CR, Dolejškova 5, 182 00 Praha 8, Czech Republic; tel.: +420 266 053 441, fax: +420 286 584 695, e-mail: zbynek@it.cas.cz

[2] Ing. Jiří Plešek, CSc., Institute of Thermomechanics AS CR, Dolejškova 5, 182 00 Praha 8, Czech Republic; tel.: +420 266 053 213, fax: +420 286 584 695, e-mail: plesek@it.cas.cz

2. Various strain measures

It is possible to define an arbitrary measure of strain. The only thing that must be present in measure definitions is the deformation gradient **F** or the displacement gradient **Z**. The most known are for instance the Green-Lagrange strain tensor

$$\mathbf{e} = \tfrac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right) = \tfrac{1}{2}\left(\mathbf{Z} + \mathbf{Z}^{\mathrm{T}} + \mathbf{Z}^{\mathrm{T}}\mathbf{Z}\right), \tag{2}$$

where **I** is the identity tensor, the infinitesimal strain tensor

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left(\mathbf{Z} + \mathbf{Z}^{\mathrm{T}}\right) \tag{3}$$

and the Lagrangean Hencky strain tensor, denoted also as right Hencky strain tensor, discussed in (Plešek & Kruisová, 2006),

$$\mathbf{h} = \tfrac{1}{2}\ln\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right). \tag{4}$$

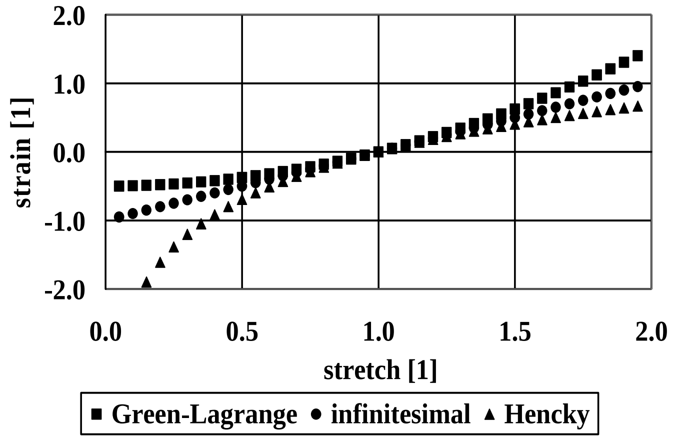

Figure 1 Various strain measures – 1D tensile-compression test

The behaviour of these quantities in the dependence on the stretch for the one dimensional tensile-compression test is depicted in the Figure 1. The stretch is greater than one for tensile and less than one for compression. Hypothetically, when equal to zero, the material is totally compressed, and its volume is equal to zero. The stretch can never gain value less than zero.

The linear character of the infinitesimal strain tensor is well perceptible. It is the linear part of the Green-Lagrange strain tensor as arised from equations (2) and (3).

However, the Green-Lagrange strain tensor is quite suitable to describe large rotations and the right Hencky strain tensor due to its logarithmic character seems to be suitable for the large displacements and large stretches description.

3. Derivation of a general energy conjugate stress tensor with respect to an arbitrary Lagrangean strain tensor

As mentioned above, the second Piola-Kirchhoff stress **S** and the Green-Lagrange strain tensor **e** are energy conjugate. Therefore, the equality of conjugation has to obey

$$\int_{V_0} \mathbf{S} : \dot{\mathbf{e}}\, \mathrm{d}V_0 = \int_{V_0} \boldsymbol{\Sigma} : \dot{\mathbf{E}}\, \mathrm{d}V_0 , \tag{5}$$

satisfying all mathematical conditions also

$$\mathbf{S} : \dot{\mathbf{e}} = \mathbf{\Sigma} : \dot{\mathbf{E}} . \tag{6}$$

Defining the Cauchy-Green strain tensor as

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \mathbf{F} , \tag{7}$$

a meaningful relation of time rates between the Green-Lagrange and the Cauchy-Green strain tensor is of form

$$\dot{\mathbf{e}} = \tfrac{1}{2} \dot{\mathbf{C}} \tag{8}$$

and the formulation (6) changes into

$$\tfrac{1}{2} \mathbf{S} : \dot{\mathbf{C}} = \mathbf{\Sigma} : \dot{\mathbf{E}} . \tag{9}$$

The Cauchy-Green strain tensor can be spectrally decomposed

$$\mathbf{C} = \mathbf{\Phi} \mathbf{\Lambda}^{2} \mathbf{\Phi}^{\mathrm{T}} \tag{10}$$

and its time rate is of form

$$\dot{\mathbf{C}} = \dot{\mathbf{\Phi}} \mathbf{\Lambda}^{2} \mathbf{\Phi}^{\mathrm{T}} + \mathbf{\Phi} 2 \mathbf{\Lambda} \dot{\mathbf{\Lambda}} \mathbf{\Phi}^{\mathrm{T}} + \mathbf{\Phi} \mathbf{\Lambda}^{2} \dot{\mathbf{\Phi}}^{\mathrm{T}} , \tag{11}$$

in eigen components

$$\dot{\mathbf{C}}' = \mathbf{\Phi}^{\mathrm{T}} \dot{\mathbf{C}} \mathbf{\Phi} = \mathbf{\Phi}^{\mathrm{T}} \dot{\mathbf{\Phi}} \mathbf{\Lambda}^{2} + 2 \mathbf{\Lambda} \dot{\mathbf{\Lambda}} + \mathbf{\Lambda}^{2} \dot{\mathbf{\Phi}}^{\mathrm{T}} \mathbf{\Phi} . \tag{12}$$

It is evident that symmetric $\dot{\mathbf{\Lambda}}$ and skewsymmetric $\dot{\mathbf{Q}}$ second order tensor can be established

$$\dot{\mathbf{\Lambda}} = \begin{bmatrix} \dot{\lambda}_1 & 0 & 0 \\ 0 & \dot{\lambda}_2 & 0 \\ 0 & 0 & \dot{\lambda}_3 \end{bmatrix} , \quad \dot{\mathbf{Q}} = \mathbf{\Phi}^{\mathrm{T}} \dot{\mathbf{\Phi}} = \begin{bmatrix} 0 & -\dot{Q}_3 & \dot{Q}_2 \\ \dot{Q}_3 & 0 & -\dot{Q}_1 \\ -\dot{Q}_2 & \dot{Q}_1 & 0 \end{bmatrix} . \tag{13}$$

Following system of equation yields from (12) and (13)

$$\begin{gathered} \dot{\lambda}_1 = \frac{\dot{C}_{11}'}{2\lambda_1} , \quad \dot{\lambda}_2 = \frac{\dot{C}_{22}'}{2\lambda_2} , \quad \dot{\lambda}_3 = \frac{\dot{C}_{33}'}{2\lambda_3} , \\ \dot{Q}_1 = \frac{\dot{C}_{23}'}{\lambda_2^2 - \lambda_3^2} , \quad \dot{Q}_2 = \frac{\dot{C}_{13}'}{\lambda_3^2 - \lambda_1^2} , \quad \dot{Q}_3 = \frac{\dot{C}_{12}'}{\lambda_1^2 - \lambda_2^2} . \end{gathered} \tag{14}$$

Analogous to the relation (10) a general strain tensor can be also spectrally decomposed

$$\mathbf{E} = \mathbf{\Phi} \, \mathrm{f}(\mathbf{\Lambda}) \mathbf{\Phi}^{\mathrm{T}} , \tag{15}$$

where $\mathrm{f}(\mathbf{\Lambda})$ is an arbitrary function of principal stretches suitable for a considered type of deformation. The rate of a general strain tensor in generalized and in eigen components is of form

$$\dot{\mathbf{E}} = \dot{\mathbf{\Phi}}\, \mathrm{f}(\mathbf{\Lambda})\mathbf{\Phi}^{\mathrm{T}} + \mathbf{\Phi}\, \dot{\mathrm{f}}(\mathbf{\Lambda})\dot{\mathbf{\Lambda}}\mathbf{\Phi}^{\mathrm{T}} + \mathbf{\Phi}\, \mathrm{f}(\mathbf{\Lambda})\dot{\mathbf{\Phi}}^{\mathrm{T}}, \tag{16}$$

$$\dot{\mathbf{E}}' = \mathbf{\Phi}^{\mathrm{T}}\dot{\mathbf{E}}\mathbf{\Phi} = \mathbf{\Phi}^{\mathrm{T}}\dot{\mathbf{\Phi}}\, \mathrm{f}(\mathbf{\Lambda}) + \dot{\mathrm{f}}(\mathbf{\Lambda})\dot{\mathbf{\Lambda}} + \mathrm{f}(\mathbf{\Lambda})\dot{\mathbf{\Phi}}^{\mathrm{T}}\mathbf{\Phi}\,. \tag{17}$$

According to (15), (16) and (17) an analogical system to (14) is obtained

$$\begin{gathered} \dot{\lambda}_1 = \frac{\dot{E}_{11}'}{\dot{\mathrm{f}}(\lambda_1)}, \quad \dot{\lambda}_2 = \frac{\dot{E}_{22}'}{\dot{\mathrm{f}}(\lambda_2)}, \quad \dot{\lambda}_3 = \frac{\dot{E}_{33}'}{\dot{\mathrm{f}}(\lambda_3)}, \\ \dot{Q}_1 = \frac{\dot{E}_{23}'}{\mathrm{f}(\lambda_2) - \mathrm{f}(\lambda_3)}, \quad \dot{Q}_2 = \frac{\dot{E}_{13}'}{\mathrm{f}(\lambda_3) - \mathrm{f}(\lambda_1)}, \quad \dot{Q}_3 = \frac{\dot{E}_{12}'}{\mathrm{f}(\lambda_1) - \mathrm{f}(\lambda_2)}. \end{gathered} \tag{18}$$

Comparing systems (14) and (18), the transformation relation between the Cauchy-Green and a general strain tensor is derived (Hrubý, 2006)

$$\begin{gathered} \dot{E}_{11}' = \dot{\mathrm{f}}(\lambda_1)\frac{\dot{C}_{11}'}{2\lambda_1}, \quad \dot{E}_{12}' = [\mathrm{f}(\lambda_1) - \mathrm{f}(\lambda_2)]\frac{\dot{C}_{12}'}{\lambda_1^2 - \lambda_2^2}, \\ \dot{E}_{22}' = \dot{\mathrm{f}}(\lambda_2)\frac{\dot{C}_{22}'}{2\lambda_2}, \quad \dot{E}_{23}' = [\mathrm{f}(\lambda_2) - \mathrm{f}(\lambda_3)]\frac{\dot{C}_{23}'}{\lambda_2^2 - \lambda_3^2}, \\ \dot{E}_{33}' = \dot{\mathrm{f}}(\lambda_3)\frac{\dot{C}_{33}'}{2\lambda_3}, \quad \dot{E}_{13}' = [\mathrm{f}(\lambda_3) - \mathrm{f}(\lambda_1)]\frac{\dot{C}_{13}'}{\lambda_3^2 - \lambda_1^2}. \end{gathered} \tag{19}$$

Substituting from (19), the equation (9) is modified

$$\begin{gathered} \tfrac{1}{2} S_{ij}'\dot{C}_{ij}' = \Sigma_{ij}'\frac{\dot{\mathrm{f}}(\lambda_i)}{2\lambda_j}\dot{C}_{ij}' \ldots\ldots\ldots\ldots\ldots i = j, \\ \tfrac{1}{2} S_{ij}'\dot{C}_{ij}' = \Sigma_{ij}'\frac{\mathrm{f}(\lambda_i) - \mathrm{f}(\lambda_j)}{\lambda_i^2 - \lambda_j^2}\dot{C}_{ij}' \ldots\ldots i \neq j. \end{gathered} \tag{20}$$

Finally, assuming the generalized components instead of the eigen ones, the relations for a general energy conjugate measure of stress are obtained

$$\begin{gathered} \Sigma_{ij} = \frac{\lambda_i}{\dot{\mathrm{f}}(\lambda_j)} S_{ij} \ldots\ldots\ldots\ldots\ldots\ldots i = j, \\ \Sigma_{ij} = \frac{\lambda_i^2 - \lambda_j^2}{2[\mathrm{f}(\lambda_i) - \mathrm{f}(\lambda_j)]} S_{ij} \ldots\ldots i \neq j. \end{gathered} \tag{21}$$

The disadvantage of the presented derivation is that the final form (21) can be expressed in tensor components only and that the second Piola-Kirchhoff stress tensor is necessary to be determined. Nevertheless, the universal character of relation (21) is indisputable.

4. Conclusion

An attempt at the derivation of explicit Cartesian components of a general energy conjugate stress tensor with respect to an arbitrary Lagrangean strain tensor has been made. Such a couple of stress and strain measures should enter any constitutive model in non-linear continuum mechanics based on either the total or the updated Lagrangean formulation. The measures of strain are expressed in the referential configuration in these cases. Constitutive equations cast in the framework of Eulerian description may be obtained via the Doyle-Ericksen formula (Doyle & Ericksen, 1956).

5. Acknowledgements

The paper has been supported by grant project of the Czech Science Foundation GAČR 101/06/0914.

6. Bibliography

Doyle, T.C. & Ericksen, J.L. (1956) Non-linear elasticity. *Advances in Applied Mechanics*, IV, Academic Press, Inc., pp. 53–115.

Hrubý, Z. (2006) General energy conjugate measures of stress and strain in continuum mechanics, In: *Proc. 6th Summer Workshop of Applied Mechanics* (M.Daniel & T.Mareš eds.), CTU FME, Prague, pp. 57-60.

Khan, A.S. & Huang, S. (1995) *Continuum Theory of Plasticity*. John Wiley & Sons, Inc., New York.

Plešek, J. & Kruisová, A. (2006) Formulation, validation and numerical procedures for Hencky's elasticity model. *Int. Journal of Computers and Structures*, 84, 17-18, pp. 1141-1150.