National Conference with International Participation
ENGINEERING MECHANICS 2006
Svratka, Czech Republic, May 15 – 18, 2006
paper no. **344**

POSITION CONTROL OF ROBOT UNDER ENDPOINT CONSTRAINTS

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Summary: *In this contribution we study PD-control of the robot endpoint that is constrained to move on a given surface described by a scalar function. This function depends on the Cartesian coordinates which are expressed in the inertial reference frame. The contact friction force arises in the direction against the motion velocity. The discussion of this contribution leads to formulate a theorem about asymptotic stability in a neighborhood of desired position.*

1. Introduction

The research of robot control is oriented to build robots that can manipulate objects. The robots are non-linear systems controlled using non-linear controllers will be used. The problem is that non-linear control methods are more difficult to apply than linear ones. Very often it is used linear controllers to control non-linear systems, firstly PD or PID controllers. To adjust the controller parameters some method of linearization may be used. Theorists often use stabilization by Lyapunov theory especially the LaSalle invariant theorem. This approach is used in this paper to control of end point of robot on a defined surface. Equations of robot motion are described by the set of Lagrange equations in the special form. The robot endpoint is constrained to move on the surface. We assume that the working space is closed and bounded, that is compact. This assumption is fulfilled in each real case.

2. Controlled system description

Let the working space Ω of robot be compact in the Euclidean space E_3 . Let L be a surface in E_3 , bounded and closed. Hence L and Ω are closed and so their intersection $L \cap \Omega$ is the closed subset of compact set Ω. Thereupon $L \cap \Omega$ is compact too. Suppose that the robot dynamics is described by non-linear equation of motion in the matrix form

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \mathbf{Q} \tag{1}$$

where $\mathbf{M}(\mathbf{q})$ is the symmetric inertial matrix, positive definite and continuous whose second partial derivatives are continuous too; $\boldsymbol{q} = (q_1,\ldots,q_n)^T$ is a set of generalized co-ordinates complete, independent which has continuous second partial derivatives; $\mathbf{g}$ denotes the gravity

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force vector

$$\mathbf{g}(\mathbf{q}) = (\partial V / \partial q_1, ..., \partial V / \partial q_n)^T ,$$

where V is the potential energy, **C** is the matrix defined by

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \left[\frac{1}{2} \dot{\mathbf{M}}(\mathbf{q}) + \mathbf{S}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{M}_0 \right].$$

where the matrix $\mathbf{M}_0$ is diagonal non-negative and usually represents damping factors while $\boldsymbol{S}$ is the skew symmetric matrix

$$S_{i,j}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \left[\sum_{k=1}^{n} \dot{q}_k \frac{\partial M_{i,k}}{\partial q_j} - \sum_{k=1}^{n} \dot{q}_k \frac{\partial M_{jk}}{\partial q_i} \right] .$$

Q on the right hand side of (1) is the vector of generalized forces (torques). Let us make a remark $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ represents Coriolis and centrifugal forces.

Define a vector $\mathbf{x} = (x_1, x_2, x_3)^T$ in the Cartesian coordinates. Let the surface L be described by the equation R(**x**) = 0, that is L={**x**; R(**x**) = 0}. Suppose that this surface is regular, so that its gradient $\partial R / \partial \mathbf{x}$ exists and is non-zero. We suppose that R has continuous second derivatives. If these conditions are fulfilled in a given neighborhoods around each point, we can define the set Ω, such that they are verified in Ω. Therefore $\boldsymbol{x} = \boldsymbol{x}(\mathbf{q})$ can be replaced into R(**x**(**q**)) which leads one to consider R(**x**(**q**)) = 0. We shall usually write **R**(**q**) instead of **R**(**x**(**q**)). The vector on the right-hand side of (1) has structure

$$\mathbf{Q} = \mathbf{u} + \mathbf{Q}_n - \mathbf{Q}_t \qquad (2)$$

where **u** is the control vector. The contact force in the normal direction of the surface is described (3)

$$\mathbf{Q}_n = \left(\mathbf{n} \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \right)^T \cdot F(\mathbf{q}) , \qquad (3)$$

where **n** is the normal vector of the surface L

$$\mathbf{n} = (\partial R / \partial \mathbf{x}) / \| \partial R / \partial \mathbf{x} \| .$$

F is a scalar function of **q** and represents the force in the orthogonal direction. The tangent force represents contact friction and is given by

$$\mathbf{Q}_t = \lambda(\dot{\mathbf{x}}) \left(\frac{\partial \mathbf{x}}{\partial \mathbf{q}} \right)^T \cdot \dot{\mathbf{x}} \qquad (4)$$

The function λ is positive, for linear dependence of the friction is only a constant. More generally λ represents more complicated dependences of friction on the motion velocity and sometimes may represent the influences of changes of position, so that very generally $\lambda = \lambda(\dot{\mathbf{x}}, \mathbf{x})$ or

$$\lambda = \lambda((\partial \mathbf{x} / \partial \mathbf{q}) \, \dot{\mathbf{q}}, \mathbf{x}(\mathbf{q})).$$

Remark: *The equation (1) is derived from Lagrange equations of Analytical mechanics and it is well known that generalized forces expresses as follows :*

$$Q_j = \sum_{i=1}^{n} \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j},$$

where $\mathbf{F}_i$ is the sum of forces that actuate onto the mass point with a position $\mathbf{r}_i$. One of these points is the end one, which we want to control on the surface (e.g. end point of the tool). The forces can be splitted into normal and tangent forces. If we denote this end point in the Cartesian coordinates as $\mathbf{x} = (x_1,x_2,x_3)^T$, we obtain relations (3) and (4).

Remember, that by definition

$$\frac{\partial \mathbf{x}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1}, & \frac{\partial x_1}{\partial q_2},\dots & \frac{\partial x_1}{\partial q_n} \\ \frac{\partial x_2}{\partial q_1}, & \dots\dots & \frac{\partial x_2}{\partial q_n} \\ \frac{\partial x_3}{\partial q_1}, & \dots\dots & \frac{\partial x_3}{\partial q_n} \end{bmatrix} \quad \text{and} \quad \frac{\partial R}{\partial \mathbf{x}} = \left[\frac{\partial R}{\partial x_1}, \ \frac{\partial R}{\partial x_2}, \ \frac{\partial R}{\partial x_3}\right].$$

Suppose that the first of them is a full rank matrix.

3. Stability and control

In this section will be treated the stability problem for PD controller with compensation of gravity force and contact force. Let the feedback be defined by

$$\mathbf{u} = -\mathbf{A}\Delta\mathbf{q} - \mathbf{B}\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) - \mathbf{Q}_{n,d}, \tag{5}$$

where **A** and **B** are diagonal positive definite matrices, which represent P and D components of PD controller,

$$\mathbf{Q}_{n,d} = \left(\mathbf{n}(\mathbf{q})\frac{\partial \mathbf{x}(\mathbf{q})}{\partial \mathbf{q}}\right)^T \cdot F(\mathbf{q}_d),$$

where $\mathbf{q_d}$ is a given desired end-position and

$$\Delta\mathbf{q} = \mathbf{q} - \mathbf{q}_d.$$

Substituting (5) into (1) yields the equation of the closed loop

$$\mathbf{M}\ddot{\mathbf{q}} + [(\mathbf{M}_0 + \mathbf{B}) + \frac{1}{2}\dot{\mathbf{M}} + \mathbf{S}]\dot{\mathbf{q}} + \mathbf{A}\cdot\Delta\mathbf{q} + \mathbf{Q}_\mathbf{t} = \mathbf{Q}_n - \mathbf{Q}_{n,d}$$

Let us define

$$\mathbf{D} = \mathbf{M}_0 + \mathbf{B} \quad \text{and} \quad \Delta F = F(\mathbf{q}) - F_d,$$

where $F(\mathbf{q}_d)=F_d$ is the desired value of the normal force at the endpoint. From the definition of the surface it can be verified that

$$\dot{\mathbf{x}}(\mathbf{q}) = \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \dot{\mathbf{q}},$$

we can rewrite the closed loop equation into

$$\mathbf{M}\ddot{\mathbf{q}} + \left[\mathbf{D} + \frac{1}{2}\dot{\mathbf{M}} + \mathbf{S} + \lambda \left(\frac{\partial \mathbf{x}}{\partial \mathbf{q}} \right)^T \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \right] \cdot \dot{\mathbf{q}} + \mathbf{A} \cdot \Delta \mathbf{q} = \left(\mathbf{n} \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \right)^T \cdot \Delta F \tag{6}$$

Let us define the following Lyapunov function

$$W(\Delta \mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} + \frac{1}{2} \Delta \mathbf{q}^T \cdot \mathbf{A} \cdot \Delta \mathbf{q} . \tag{7}$$

If we differentiate this function and using

$$\frac{d}{dt}(\mathbf{q} - \mathbf{q}_d) = \dot{\mathbf{q}},$$

we get :

$$\frac{dW(\Delta \mathbf{q}, \dot{\mathbf{q}})}{dt} = \dot{\mathbf{q}}^T (\mathbf{M}\ddot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{M}}\dot{\mathbf{q}} + \mathbf{A} \cdot \Delta \mathbf{q}) .$$

Substituting this equation into (6), then

$$\frac{dW}{dt} = -\dot{\mathbf{q}}^T (\mathbf{D} + \mathbf{S})\dot{\mathbf{q}} - \lambda \left(\frac{\partial \mathbf{x}}{\partial \mathbf{q}} \dot{\mathbf{q}} \right)^T \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \dot{\mathbf{q}} + \left(\mathbf{n} \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \dot{\mathbf{q}} \right)^T \cdot \Delta F \tag{8}$$

The matrix **S** is skew symmetric and hence

$$\dot{\mathbf{q}}^{\mathbf{T}} \mathbf{S} \dot{\mathbf{q}} = 0 .$$

For points of the surface R(**x**)=0 by differentiation

$$\frac{\partial R}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \dot{\mathbf{q}} = 0$$

and hence

$$\mathbf{n} \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \dot{\mathbf{q}} = 0 . \tag{9}$$

From these results we obtain the inequality

$$\frac{dW}{dt} = -\dot{\mathbf{q}}^T \mathbf{D} \dot{\mathbf{q}} - \lambda \left(\frac{\partial \mathbf{x}}{\partial \mathbf{q}} \dot{\mathbf{q}} \right)^T \cdot \left(\frac{\partial \mathbf{x}}{\partial \mathbf{q}} \dot{\mathbf{q}} \right) \leq 0 . \tag{10}$$

The matrix **D** is diagonal positive definite and the function λ is positive. Hence the relations (7) and (10) show that the control process is stable, as it is well known from the Lyapunov theory of stability.

From (9) we can derive

$$\mathbf{n}\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\ddot{\mathbf{q}} = -\frac{d}{dt}\left(\mathbf{n}\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right)\dot{\mathbf{q}} \,. \tag{11}$$

If we use (11) in (6), we can compute the difference ΔF as a function of $\mathbf{q}, \dot{\mathbf{q}}$. Because all functions and matrices are continuous in compact working space, they are bounded and hence if $\dot{\mathbf{q}}$ and $\Delta\mathbf{q}$ converge to zero, then ΔF converges to zero too. ΔF is obtained from (6) and (11) :

$$\Delta F = \left\{\mathbf{KM}^{-1}\mathbf{F}\cdot\dot{\mathbf{q}} - \dot{\mathbf{K}}\dot{\mathbf{q}} + \mathbf{KM}^{-1}\mathbf{A}\cdot\Delta\mathbf{q}\right\} / \mathbf{KM}^{-1}\mathbf{K}^{\mathbf{T}} \tag{12}$$

where

$$\mathbf{K} = \mathbf{n}^{T}(\mathbf{q})\frac{\partial \mathbf{x}}{\partial \mathbf{q}}, \quad \mathbf{N} = \frac{\partial \mathbf{x}}{\partial \mathbf{q}},$$

$$\mathbf{F} = \mathbf{D} + 0.5\dot{\mathbf{H}} + \mathbf{S} + \lambda\mathbf{N}^{\mathbf{T}}\mathbf{N} \,.$$

Consider the inequality

$$W(\Delta\mathbf{q}, \dot{\mathbf{q}}) < \eta \,.$$

If $\eta \to 0^{+}$, then from (7), with respect that the matrix **M** has a positive lower bound, follows $\mathbf{q} \to \mathbf{q_d}$, $\dot{\mathbf{q}} \to \mathbf{0}$ and hence from (12) $\Delta F \to 0$. Therefore for $F_d > 0$, there exists $\alpha > 0$, such that for arbitrary $\eta < \alpha$ $|\Delta F| < F_d$. But the inequality $|\Delta F| < F_d$ and $F_d > 0$ is equivalent to F>0 and the endpoint is in contact with the surface, that is R(**x**(**q**))=0. Therefore, if the control process will be started from any initial point

$$(\mathbf{q}(0), \dot{\mathbf{q}}(0)) \,,$$

such that

$$W(\Delta\mathbf{q}(0), \dot{\mathbf{q}}(0)) < \eta \,,$$

then, as the function W is not increasing (10), it is clear that

$$W(\Delta\mathbf{q}(t), \dot{\mathbf{q}}(t)) < \eta$$

for any t >0. According to previous discussion 0 < F, and hence the endpoint will be maintained in contact with the constraint surface.

Accordingly, from (7) it follows that there is a neighborhood of the point ($\mathbf{q}_d$,**0**) such that every trajectory which starts from this neighborhood will remain there for every positive t.

Next, consider the invariant set of equation (6). According to (10), every point from the invariant set must be (**q, 0**) and must satisfy the equation

$$\mathbf{A}\cdot\Delta\mathbf{q} = \left(\mathbf{n}\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right)^{T}\cdot\Delta F \,. \tag{13}$$

As we supposed, the second partial derivatives of R(**x**) exist and are continuous. The vector function

$$\mathbf{K}(\mathbf{q}) = \mathbf{n}(\mathbf{q})\frac{\partial \mathbf{x}}{\partial \mathbf{q}}$$

has continuous partial derivatives and using the middle value theorem of functions in linear norm spaces it follows that

$$\left|\mathbf{K}(\mathbf{q})\Delta\mathbf{q}-\mathbf{K}(\mathbf{q_d})\Delta\mathbf{q}\right| \le \left\|\mathbf{K}(\mathbf{q})-\mathbf{K}(\mathbf{q_d})\right\|\cdot\left\|\Delta\mathbf{q}\right\| \le \sup_{\mathbf{p}\in[\mathbf{q},\mathbf{q}_d]}\left\|\dot{\mathbf{K}}(\mathbf{q})_{\mathbf{q}_d}\right\|\cdot\left\|\Delta\mathbf{q}\right\|^2 \le C_1\left\|\Delta\mathbf{q}\right\|^2$$

where $\mathbf{p}\in[\mathbf{q},\mathbf{q}_d]$ means, that the vector **p** is a convex combination of the points **q, q**$_d$. C_1 is a non-negative constant. From this follows

$$\left|\mathbf{K}(\mathbf{q})\Delta\mathbf{q}\right| \le \left|\mathbf{K}(\mathbf{q_d})\Delta\mathbf{q}\right| + C_1\left\|\Delta\mathbf{q}\right\|^2 . \tag{14}$$

If we use Taylor's expansion theorem at point **q**$_d$ of function R(**x**(**q**)), then

$$R(\mathbf{q}) = R(\mathbf{d}_d) + \left(\frac{\partial R}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right)_{\mathbf{q}=\mathbf{q}_d}\cdot\Delta\mathbf{q} + \frac{1}{2}\Delta\mathbf{q}^T\mathbf{H}\Delta\mathbf{q} + o\left(\left\|\Delta\mathbf{q}\right\|^2\right) \tag{15}$$

If points **q, q**$_d$ are on the surface L, then R(**x**(**q**))=0, R(**x**(**q**$_d$))=0, so it follows from (15) that

$$\left(\frac{\partial R}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right)_{\mathbf{q}_d}\cdot\Delta\mathbf{q} = -\frac{1}{2}\Delta\mathbf{q}^T\mathbf{H}\Delta\mathbf{q} + o\left(\left\|\Delta\mathbf{q}\right\|^2\right) \tag{16}$$

where **H** is the Hessian matrix. The non-zero function

$$\left\|\partial R/\partial\mathbf{x}\right\|$$

is continuous on the compact set Ω or Ω∩L, respectively. Hence there exists a positive m, such that

$$m = \inf_{\mathbf{x}}\left\|\partial R/\partial\mathbf{x}\right\| .$$

If we divide (16) by $\left\|\partial R/\partial\mathbf{x}\right\|_{\mathbf{x}(\mathbf{q}_d)}$, then we obtain

$$\left|\mathbf{K}(\mathbf{q}_d)\cdot\Delta\mathbf{q}\right| = \left|\left(\mathbf{n}\frac{\partial\mathbf{x}}{\partial\mathbf{q}}\right)_{\mathbf{q}_d}\cdot\Delta\mathbf{q}\right| \le \left(\frac{1}{2}\left\|\Delta\mathbf{q}\right\|^2\cdot\left\|\mathbf{H}\right\| + o\left(\left\|\Delta\mathbf{q}\right\|^2\right)\right)m^{-1} \le C_2\left\|\Delta\mathbf{q}\right\|^2$$

From this inequality, and using (13) and (14) we get

$$0 = \Delta\mathbf{q}^{\mathbf{T}}\left[\mathbf{A}\cdot\Delta\mathbf{q} - \mathbf{K}^T\cdot\Delta F\right] = \Delta\mathbf{q}^{\mathbf{T}}\mathbf{A}\Delta\mathbf{q} - (\mathbf{K}\Delta\mathbf{q})^T\Delta F \ge \Delta\mathbf{q}^{\mathbf{T}}\mathbf{A}\Delta\mathbf{q} - (C_1 + C_2\Delta F)\left\|\Delta\mathbf{q}\right\|^2$$
$$\ge (C_3 - C_1 - C_2\Delta F)\left\|\Delta\mathbf{q}\right\|^2$$

for positive constant C_3. But for suitable neighborhood of the end point it is possible to choose C_3 sufficiently large such that

$$C_3 - C_1 - C_2\Delta F > 0 . \tag{17}$$

and hence Δ**q** = 0. These results demonstrate that the maximum invariant set of (6) has only one point (**q**$_d$,**0**). If we apply the LaSalle invariant theorem, then

$$\mathbf{q}(t) \to \mathbf{q}_d, \dot{\mathbf{q}}(t) \to \mathbf{0} \text{ as } t \to \infty .$$

But this means that $F \to F_d$ as $t \to \infty$.

Remark that often it is possible to choose the matrix **A** sufficiently large so that C_3 is sufficiently large, too.

4. Conclusion

Let us assume that $R(\mathbf{q}_d) = 0$, $\mathbf{q}_d \in \Omega$, $F_d > 0$,

$$\partial R / \partial \mathbf{q} \neq 0 \text{ in } \Omega,$$

and the diagonal matrix $\mathbf{A} > \mathbf{0}$ is chosen sufficiently large. Then the equilibrium point $(\mathbf{q}_d,\mathbf{0})$ is asymptotically stable in the sense that there exists a neighborhood B_d of $(\mathbf{q}_d,\mathbf{0})$ on the subsurface $L \cap \Omega$, such that every solution of (6) starting from arbitrary initial state

$$(\mathbf{q}(0), \dot{\mathbf{q}}(0)) \in B_d ,$$

asymptotically approaches the point $(\mathbf{q}_d,\mathbf{0})$ as $t \to \infty$ and $F \to F_d$ as $t \to \infty$.

The controller parameters can be adjusted by optimization of suitable criterion. The controlled system is nonlinear and hence for optimization of parameters it is interesting to combine global optimization methods with local ones, which are well known in mathematical programming.

5. References

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