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ON APPLICATIONS OF GENERALIZED FUNCTIONS TO CALCULATIONS OF BEAM DESIGN ELEMENTS

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Summary: *The paper deals with the mathematical representation of discontinuities in loading and geometry of beams under the bending. A system of ordinary differential equations (SODE) that makes the classical model of the beam bending does not hold at points of a discontinuity of functions the shear force, the bending moment, the slope, and the deflection for their derivatives are not defined at points of jump discontinuities. To solve this problem we use the generalized functions (distributions) and we generalize the SODE of the beam bending so as to hold at loading and geometry discontinuity points. Further we show essential formulae for looking for solutions to the generalized SODE using the direct integration without employing the Laplace transform.*

1. Introduction

The beams with discontinuities in loading or geometry are usually calculated in such a way that they are at first divided into finite elements without the discontinuities. Then continuous solutions with integration constants are determined for every such element apart. Finally the integration constants are determined from boundary and continuity conditions.

When we use the Dirac distribution and the Heaviside's unit step function the generalized beam bending model with discontinuities in loading and geometry can be derived and solved like one differential problem without continuity conditions. Such approach is presented in this paper.

2. The Scope of the Validity of the Beam Bending Classical Mathematical Model

A system of ordinary differential equations (SODE) describing the plane bending of a slender beam according to the Bernoulli-Navier hypothesis is composed of four first order differential equations as follows

$$\frac{d}{dx}\mathrm{T}(x) = -q_n(x) \tag{1}$$

$$\frac{d}{dx}\mathrm{M}(x) = \mathrm{T}(x) + q_m(x) \tag{2}$$

$$\frac{d}{dx}\phi(x) = \frac{\mathrm{M}(x)}{E\,\mathrm{J}(x)} \tag{3}$$

$$\frac{d}{dx}\mathrm{v}(x) = \phi(x) \tag{4}$$

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where

T(x)	shear force [*N*],
M(x)	bending moment [*Nm*],
$\phi(x)$	slope [*rad*],
$v(x)$	deflection [*m*] (positive direction upward),
$q_n(x)$	equivalent distributed force [*N/m*],
$q_m(x)$	equivalent distributed moment [*Nm/m*],
J(x)	area moment of inertia [m^4],
E	Young's modulus [*Pa*].

This SODE is valid in the space of real functions if all the derivatives shown exist. However if any of the functions has a jump discontinuity, which is common in calculating experience, then it has not the derivative and the SODE does not hold at the point of discontinuity.

If we replace the space of real functions by the space of distributions then the derivative of the generalized function with jump discontinuities is defined as the distributional derivative (Kanwal, 2004) and is expressed for a function f(x) that has one point of the jump discontinuity by equation

$$f' = \{f'(x)\} + [f]_{x_0} \,.\, \delta(x - x_0) \tag{5}$$

where

$\{f'(x)\}$	classical derivative,
$[f]_{x_0}$ = f(x_0 + 0) - f(x_0 - 0)	magnitude of the jump discontinuity at $x = x_0$,
$\delta(x - x_0)$	Dirac distribution (Dirac-delta function or unit-impulse function) at $x = x_0$.

3. The Distributional Derivative of the Functions the Shear Force, the Bending Moment, the Slope, and the Deflection with One Jump Discontinuity

Let the shear force T(x) have a jump discontinuity at point x = a_1 of magnitude

$$T(a_1 + 0) - T(a_1 - 0) = -F_1 \tag{6}$$

as a result of the action of a concentrated force. Then the distributional derivative (5) of T(x) is

$$T' = \{T'(x)\} + [T]_{a_1} \,.\, \delta(x - a_1) = -q_n(x) - F_1\,\delta(x - a_1) = -(q_n(x) + F_1\,\delta(x - a_1)) \,. \tag{7}$$

By analogy with the equivalent distributed force we can name the sum $q_n(x) + F_1\,\delta(x - a_1)$ as the generalized equivalent distributed force whereof the concentrated force is expressed as an impulse of magnitude F_1 acting at x = a_1 .

Let the bending moment M(x) have a jump discontinuity at x = b_1 of magnitude

$$M(b_1 + 0) - M(b_1 - 0) = M_1 \tag{8}$$

as a result of the action of a concentrated force couple. Then the distributional derivative (5) of M(x) is

$$\mathrm{M}' = \{\mathrm{M}'(\mathrm{x})\} + [M]_{b_1} \, . \, \delta(x - b_1) = \mathrm{T}(x) + q_m(x) + M_1\, \delta(x - b_1)\, . \qquad (9)$$

By analogy with the equivalent distributed moment we can name the sum $q_m(x) + M_1\, \delta(x - b_1)$ as the generalized equivalent distributed moment whereof a concentrated force couple is expressed as an impulse of magnitude M_1 acting at $x = b_1$.

Let the slope $\phi(x)$ have a jump discontinuity at $x = c_1$ of magnitude

$$\phi\,(c_1 + 0) - \phi\,(c_1 - 0) = \Phi_1 \qquad (10)$$

as a result of placing of an internal hinge. Then the distributional derivative (5) of $\phi(x)$ is

$$\phi\,' = \{\,\phi\,'(\mathrm{x})\} + [\phi]_{c_1} \, . \, \delta(x - c_1) = \frac{\mathrm{M}(x)}{E\,\mathrm{J}(x)} + \Phi_1\, \delta(x - c_1)\, . \qquad (11)$$

The unknown value Φ_1 of the relative rotation of cross sections at the hinge is determined by a deformation condition of the hinge. If the internal hinge contains a rotational spring with the stiffness k_r then the deformation condition of the hinge is

$$\mathrm{M}(c_1 + 0) = k_r\, \Phi_1\, . \qquad (12)$$

When the internal hinge does not contain any rotational spring we use the equation

$$\mathrm{M}(c_1 + 0) = 0 \qquad (13)$$

as the deformation condition of the hinge. These deformation conditions hold also in the case that a concentrated force couple is connected to the hinge near to the left. However if the concentrated force couple is connected to the hinge near to the right then the limit to the left M(c_1 -0) must be used in the deformation condition of the hinge.

Let the deflection $\mathrm{v}(x)$ have a jump discontinuity at $x = d_1$ of magnitude

$$\mathrm{v}(d_1 + 0) - \mathrm{v}(d_1 - 0) = \Delta_1 \qquad (14)$$

as a result of placing of an internal slide rail. Then the distributional derivative (5) of v(x) is

$$\mathrm{v}' = \{\mathrm{v}'(\mathrm{x})\} + [v]_{d_1} \, . \, \delta\,(\mathrm{x} - d_1) = \phi(x) + \Delta_1\, \delta(x - d_1)\, . \qquad (15)$$

The unknown value Δ_1 of relative displacement of cross sections in the internal slide rail is determined by a deformation condition of the slide. If this internal slide rail contains a translational spring with stiffness k_t then the deformation condition of the slide is

$$\mathrm{T}(d_1 + 0) = k_t\,(-\Delta_1)\, . \qquad (16)$$

Because we have chosen positive direction of deflection upward the shear force T(d_1 + 0) causes a negative change in relative displacement Δ_1 in the slide. When the internal slide rail does not contain any translational spring we use the equation

$$\mathrm{T}(d_1 + 0) = 0 \qquad (17)$$

as the deformation condition for the internal slide rail. These deformation conditions hold also in the case that a concentrated force is connected to the slide rail near to the left. However if the concentrated force is connected to the slide rail near to the right then the limit to the left T(d_1 -0) must be used in the deformation condition of the slide rail.

4. The Generalized SODE of the Beam Bending for the Finite Number of Discontinuities in Loading and Geometry

Let the shear force T(x) have the jumps of the magnitude $F_1, F_2, ..., F_k$ at $a_1, a_2, ..., a_k$. Let the bending moment M(x) have the jumps of the magnitude $M_1, M_2, ..., M_l$ at $b_1, b_2, ...,$ b_l. Let the slope $\phi(x)$ have the jumps of the magnitude $\Phi_1, \Phi_2, ..., \Phi_m$ at $c_1, c_2, ..., c_m$. Let the deflection v(x) have the jumps of the magnitude $\Delta_1, \Delta_2, ..., \Delta_n$ at $d_1, d_2, ..., d_n$. Now we can generalize the distributional derivatives (7), (9), (11), (15) for the finite number of the jump discontinuities and thus we obtain the final SODE. The result is

$$\frac{d}{dx}\mathrm{T}(x) = -\left(q_n(x) + \left(\sum_{i=1}^{k} F_i\,\delta(x-a_i)\right)\right) \tag{18}$$

$$\frac{d}{dx}\mathrm{M}(x) = \mathrm{T}(x) + q_m(x) + \left(\sum_{i=1}^{l} M_i\,\delta(x-b_i)\right) \tag{19}$$

$$\frac{d}{dx}\phi(x) = \frac{\mathrm{M}(x)}{E\,\mathrm{J}(x)} + \left(\sum_{i=1}^{m} \Phi_i\,\delta(x-c_i)\right) \tag{20}$$

$$\frac{d}{dx}\mathrm{v}(x) = \phi(x) + \left(\sum_{i=1}^{n} \Delta_i\,\delta(x-d_i)\right) \tag{21}$$

5. The Solution Procedure of the Generalized SODE of the Beam Bending with the Dirac Distributions

In order to find the general solution of the generalized SODE (18) to (21) we use the direct integration of the right sides by means of the following essential formulae

$$\int \delta(x-a)\,dx = \mathrm{Heaviside}(x-a) \tag{22}$$

$$\int \mathrm{f}(x)\,\mathrm{Heaviside}(x-a)\,dx = \left(\int \mathrm{f}(x)\,dx - \int \mathrm{f}(x)\,dx\Big|_{x=a}\right)\mathrm{Heaviside}(x-a) \tag{23}$$

where Heaviside(x) stands for the Heaviside's step function. Integration constants are not shown at these formulae because these integrals are only fractional results and are not the general solution of a certain differential equation of the SODE.

As an example we employ the formula (23) with f(x) = x^k , where *k* is different from minus one,

$$\int x^k\,\mathrm{Heaviside}(x-a)\,dx = \left(\frac{x^{(k+1)}}{k+1} - \frac{a^{(k+1)}}{k+1}\right)\mathrm{Heaviside}(x-a)\ . \tag{24}$$

Now we show two ways to calculate the expression $\frac{\mathrm{M}(x)}{\mathrm{J}(x)}$ if J(x) or its derivative has the jump discontinuities. Let J(x) be expressed for simplicity as follows

$$\mathrm{J}(x) = \{ \begin{array}{ll} f_1(x) & x < b \\ f_2(x) & otherwise \end{array} . \tag{25}$$

Further let us assume that we have obtained the bending moment in the form

$$\mathrm{M}(x) = g_1(x) + g_2(x)\ \mathrm{Heaviside}(x - a) \tag{26}$$

by integration of Eq. (19). Let a be equal or greater than b. We shall see shortly that this assumption is not binding for our considerations but enables us to complete our calculation.

The first and simple way to calculate the fraction $\frac{\mathrm{M}(x)}{\mathrm{J}(x)}$ of discontinuous functions is that we at first convert the fraction $\frac{\mathrm{M}(x)}{\mathrm{J}(x)}$ into product of discontinuous functions

$$\frac{\mathrm{M}(x)}{\mathrm{J}(x)} = \left(\left(-\frac{1}{f_1(x)} + \frac{1}{f_2(x)} \right) \mathrm{Heaviside}(x - b) + \frac{1}{f_1(x)} \right) (g_1(x) + g_2(x)\ \mathrm{Heaviside}(x - a)) \tag{27}$$

and then so as to simplify this expression we use the following formula for product of step functions (if a>=b)

$$\mathrm{Heaviside}(x - a)\ \mathrm{Heaviside}(x - b) = \mathrm{Heaviside}(x - a)\ . \tag{28}$$

After simplification we obtain

$$\frac{\mathrm{M}(x)}{\mathrm{J}(x)} = \frac{(g_1(x) f_1(x) - g_1(x) f_2(x))\ \mathrm{Heaviside}(x - b)}{f_2(x) f_1(x)} + \frac{g_2(x)\ \mathrm{Heaviside}(x - a)}{f_2(x)} + \frac{g_1(x)}{f_1(x)} \tag{29}$$

The second way to calculate the fraction $\frac{\mathrm{M}(x)}{\mathrm{J}(x)}$ of discontinuous functions is that if we expressed J(x) as follows

$$\mathrm{J}(x) = (f_2(x) - f_1(x))\ \mathrm{Heaviside}(x - b) + f_1(x) \tag{30}$$

we would have to solve more complicated equation as follows

$$\frac{g_1(x) + g_2(x)\ \mathrm{Heaviside}(x - a)}{(f_2(x) - f_1(x))\ \mathrm{Heaviside}(x - b) + f_1(x)} = c_1(x) + c_2(x)\ \mathrm{Heaviside}(x - b) + c_3(x)\ \mathrm{Heaviside}(x - a) \tag{31}$$

Integration constants from the general solution of SODE (18) to (21) are determined by means of the boundary conditions. If we solved the SODE by means of the Laplace transform the integration constants would have the form of initial parameters T(0), M(0), ϕ (0), v(0).

6. Conclusions

The derived generalized SODE of the classical beam bending model shows that all the point discontinuities like a concentrated force, a concentrated moment, an internal hinge or internal slide can be expressed by means of the Dirac distribution. The jump discontinuities of an equivalent distributed force or equivalent distributed moment or flexural stiffness can be expressed by means of the Heaviside's unit step function.

The generalized SODE (18) to (21) for a certain beam is solved using the formulae (22) and (23) for the direct integration. The integration constants are determined from the boundary conditions. Other unknown quantities like for example the magnitude of a reaction in an internal support point or the relative slope of cross sections in an internal hinge can be calculated from the deformation conditions.

The differential problem with the generalized SODE can also be expressed and solved by means of the symbolic programming. Such technique is very effective.

The contribution of this paper is in the exact and simple derivation of the complete generalized SODE of the classical beam bending model on the basis of the distributional derivative definition. The form used for the generalized SODE of the first order has certain advantages as compared with one ordinary differential equation of the fourth order. This form does not contain derivatives of the Dirac distribution and shows the sequence in which the equations (18) to (21) should be integrated.

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