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FAST ITERATIVE SOLVERS FOR THE EFFICIENT PROBABILISTIC RELIABILITY ASSESSMENT OF SPECIAL MECHANICAL SYSTEMS BY THE SBRA METHOD

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Summary: *In the paper, we will describe our experiences with the full probabilistic reliability assessment of a concrete beam by the finite element method using simulation techniques. Special iterative solvers will be used for the efficient solution of repeated linear elasticity models arising from the reliability assessment. Moreover, the effect of mesh refinement is studied. The results of the assessment are presented and discussed.*

1. Introduction

Solving of a large linear system of equations plays important part in a mathematical modelling using numerical methods such as Finite Element approximation. Popular direct solvers, which are based on Gaussian elimination or LU-decomposition of the stiffness matrix are efficient enough for small and middle-sized problems. Unfortunately, for large problems direct methods become too expensive mainly due to the amount of storage of the elimination process. On the other hand, iterative methods, such as Conjugate Gradient methods (CG), do not involve time and memory consuming elimination process and CG-based algorithms can work very effectively in modern high-performance parallel computer environment.

The paper is organized in a following way. In Section 2, we will introduce shortly the multiple linear system of equations. In Section 3, the Successive Block CG algorithm is briefly described. Finally, an efficient solution of repeated linear elasticity models arising from the probabilistic reliability assessment of a finite element model of a structure using the simulation techniques will show the efficiency of the here presented approach. Moreover, the effect of mesh refinement is studied and discussed. This extends experiments with iterative solvers reported in [6] and [20].

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2. Multiple linear system of equations

Let us assume the following system of the linear equations with q right-hand sides (RHSs):

$$KX = F \tag{1}$$

where K is a real symmetric positive definite sparse matrix of an order n and $F = [f^1, \ldots, f^q]$ is an $n \times q$ matrix of right-hand sides. Two basic approaches for solving the linear systems of equation may be applied: direct and iterative methods. *The direct methods* are popular in commercial software systems thanks to their robustness. The algorithms of this type involve a factorization of the matrix K, which is computed only once, because the system of linear equation for all right-hand sides has the same stiffness matrix K. Unfortunately, the factorization of the matrix K is memory consuming operation because the factorized matrix becomes more dense than the original sparse matrix K. Hence, the direct methods could become impractical, especially for three-dimensional problems, where the matrix K is very large. Furthermore, the factorization of the sparse matrix is not well scalable operation so that its efficient parallel implementation is not straightforward.

On the other hand, *the iterative methods* generate a sequence of approximate solutions $\{X_k\}$. These methods involve the matrix K only in context of matrix-vector multiplication. Due to this fact, it is sufficient to store only nonzero elements of the matrix K and sophisticated storage systems for sparse matrices (for example see [12]) which saves memory requirements may be used. A usage of the iterative solvers is also popular on parallel computation architecture, see [14]. The most popular iterative method is the well-known conjugate gradient (CG) method.

This paper presents our experiences with the Successive Block CG method for solving (1) where all the right-hand sides are available simultaneously.

3. The Successive Block CG Method for multiple right-hand sides

The original Block CG algorithm [7], [11] requires to compute the inversion of the matrices $P_k^T K P_k$ and $R_k^T M^{-1} R_k$. Generally, the search direction matrix P_k and the matrix of residuum R_k may not have the full column rank, for example when the RHSs vectors are linearly dependent. If a stable version of BCG is required, the linear dependent RHSs have to be removed from the problem. Unfortunately, the norms of residuals of these linear dependent RHSs do not need to be small enough. We shall discuss now how to find smartly the solution of these unsolved RHSs.

The stable version of the BCG algorithm has to monitor the column rank of the direction matrix P_k or R_k. In particular, this matrix does not have the full column rank, the BCG method removes these linear dependent RHSs from the problem even if the corresponding residuals of these RHSs are not small enough. On the other hand, the Successive block CG method removes these dependent RHSs only from the search direction process. The corresponding solutions X_k^s are obtained by successive approach. Following [11], the preconditioned SBCG method for multiple RHS vector is described in the following way.

Algorithm SBCG [Preconditioned Successive Block CG Method]

Initialize: $k = 0; R_0 = F - KX_0;$
fill the sets m and s such that $m \cup s = \{1, \ldots, q\}$ and $m \cap s = \{\}$;
while $(\max(\|r_k^m\|/\|r_0^m\|) > \varepsilon, i = 1, \ldots, q)$ **and** $(k < k_{max})$ **do**
begin
$k = k + 1;$
solve $MZ_k = R_k^m;$
analyse $(R_{k-1}^m)^T Z_{k-1}$; move dependents RHSs to the slave sytem;
if $k = 1$ **then**
$P_1 = Z_0;$
else
$\beta_k = ((R_{k-2}^m)^T Z_{k-2})^{-1} (R_{k-1}^m)^T Z_{k-1};$
$P_k = Z_{k-1} + P_{k-1}\beta_k;$
endif.
$U_k = KP_k;$
$[\alpha_k^m, \alpha_k^s] = (P_k^T U_k)^{-1}[(R_{k-1}^m)^T Z_{k-1}, (R_{k-1}^s)^T Z_{k-1}];$
$[X_k^m, X_k^s] = [X_{k-1}^m, X_{k-1}^s] + P_k[\alpha_k^m, \alpha_k^s];$
$[R_k^m, R_k^s] = [R_{k-1}^m, R_{k-1}^s] - U_k[\alpha_k^m, \alpha_k^s];$
end.

Let us summarize this algorithm. Firstly, SBCG splits the separate RHSs into the two disjoint following sets. Initially, the set m contains the indexes of the master RHSs and the set s contains the indexes of the slaves RHSs. The RHSs corresponding to the set m are solved using the BCG algorithm. Let symbol $|m|$ denotes the number of indexes in master set m. Then the search direction P_k is $n \times |m|$ matrix. To prevent numerical instability during the search direction process, the full column rank of P_k have to be guaranteed.
Let $0 \leq coef \leq 1$ is a given constant. In our code, numerical stability of matrix $(R_{k-1}^m)^T Z_{k-1}$ is monitored by the orthogonal-triangular decomposition in the following way.

Algorithm Findep [Find and remove dependent indexes]

Orthogonal-triangular decomposition: $[q, v] = qr((R_{k-1}^m)^T Z_{k-1});$
for $i = 1$ **until** $|m|$ **do**
begin {for each master index}
$relcoef_i = |v_{ii}|/max(|v_{ii}|);$
if $relcoef_i < coef$ **then**
move $m(i)$-th index to the slave system;
endif.
end.

Of course, the most time-consuming operation of the Findep algorithm is the orthogonal-triangular decomposition. Let us accent that the size of this decomposition depends only on the number of indexes in master set m. We shall discuss the value of the coefficient $coef$ in the following section.

4. Probabilistic reliability assessment of structures

Let the resistance of the structure is expressed by the variable R and load effect by variable S. Let the safety of the structure is expressed using the safety function Z in the following way:

$$Z = R - S. \tag{2}$$

The situations where $Z < 0$ represents a failure in the structure, whereas situations $Z > 0$ are safe, see for instance [15], [16] and [17]. Of course, both variables R and S are random by nature and the equation 2 can be rewritten as

$$Z = g(X_1, X_2, \ldots, X_n). \tag{3}$$

Here symbols $X_1, X_2, \ldots, X_n$ denotes random variables, which express a rule geometrical and material characteristics, loadings and optionally effects of other factors and the symbol g denotes the performance function of the structure. For more details see for instance [15], [16] and [17]. Than probability of the failure of the structure can be formulated by the form

$$P_f = P(Z < 0) = P(g(X_1, X_2, \ldots, X_n) < 0). \tag{4}$$

The aim of the probabilistic reliability assessment leads to the reliability check expressed by

$$P_f < P_d, \tag{5}$$

where the symbol P_f denotes the calculated probability of failure and the symbol P_d denotes the target design probability P_d given in (expert) codes, see for instance [15], [16] and [17].
The equation 4 can be calculated approximately by FORM and SORM methods, see for instance [15] or directly by the simulation approach, see for instance [16] and [17] and [15].

4.1. Model description

This example was derived from the Calfem home page [23], see ”CALFEM/Pre user interface tutorial”, where it is possible to find the finite element model and its solution via Calfem toolbox, too. In this paper, we extend the original deterministic model by the case where all loads are assumed to be random variables. Moreover, the probabilistic reliability assessment of the structure will be estimated by simulation approach using direct Monte Carlo method and Importance Sampling method.
Consider the concrete frame subjected to a uniformly distributed loads F_1, F_2, ..., F_6 as shown in Fig. 1. The model has the following deterministic parameters: Young’s modulus $E = 10.5$ GPa, Poisson’s ratio $\nu = 0.15$ and thickness $t = 0.20$ m. All loads are assumed to be normal random variables with parameters as shown in Tab. 1.

The frame is discetized using the finite element code CALFEM. In order to study the effect of mesh refinement, we assumed two various finite element meshes, denoted as Geometry5 and Geometry7, see Fig. 2 and Fig. 3.

In our model, the safety function (2) was expressed in the following way. The R denoted concrete tensile strength described by normal random variable with parameters $R = 1 \pm 0.1$ MPa. The S denoted the maximum value of the main principal stress of an element of the structure. For evaluating of deterministic values of S we used modified deterministic Calfem finite element model taken from [23]. The computation of probability of failure by (4) was powered by the SBRA method, see for instance [16], [17] and [18].

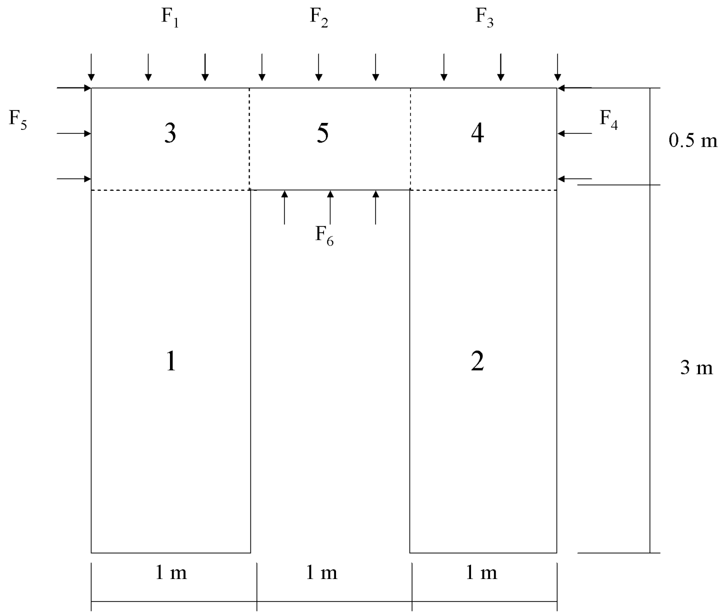

Figure 1: The model problem for the probabilistic reliability assessment. The geometry of the frame contains 5 subdomains denoted by symbols 1, 2, . . . , 5.

Table 1: Parameters of the random loads.

Variable name	Mean value	Standard deviation
F_1	15 kN	5 kN
F_2	15 kN	5 kN
F_3	15 kN	5 kN
F_4	4 kN	4 kN
F_5	4 kN	4 kN
F_6	0 kN	4 kN

4.2. Preprocessing

In order to detect low probability events on tail areas, we used the variance reduction technique based on Importance Sampling. With the current implementation, the importance sampling density function is set as the uniform distribution on the same domain as the original distribution. Numerical experiments indicated advantages of this selection when no additional information about structural behaviour is available. For implementation details see [22] and [19]. Moreover, the same probabilistic reliability assessment problem was solved also by the direct Monte Carlo simulation.
Because of the fact that we assume stochastic character of loads in our model, the stochastic contribution of random loads will influence only the right hand side vectors of the linear system of equations.

4.3. Processing

When the FEM mesh Geometry5 was applied, 1 000 simulation steps were computed, so the corresponding multiple system of linear equations had 1 000 right hand sides and the total num-

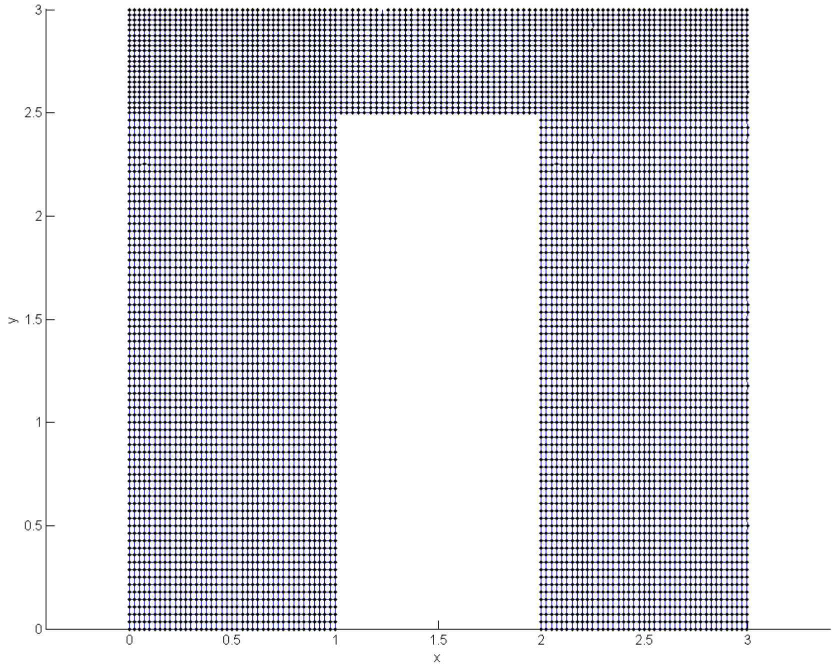

Figure 2: The finite element mesh of the frame using CALFEM (Problem name: FEM-Frame2_Geometry5).

ber of unknowns was $16\ 188 \times 1\ 000$, see Tab. 2.
As the solver of this multiple linear system of equations we used the SBCG algorithm, where the $coef$ was selected as a parameter, see Tab. 3.
Analysing Tab. 3 we can see, that the $coeq = 10^{-6}$ is the best choice for our case. For example, in order to solve 1 000 right hand sides using the SCG method, 1 424.75 sec. were needed, while the "favored" SBCG algorithm with $coeq = 10^{-6}$ needed only 546.578 sec. The SBCG relative speed-up was $1\ 424.75/546.578 \approx 2.6$. On the other hand, the number of matrix-vector operation was smaller when SCG algorithm was applied. To solve all 1 000 linear systems of equations, only 446 matric-vector operations were needed. Let us notice that the solution of *one* linear system by the classical PCG required 205 matric-vector operations. When the finer FEM mesh Geometry7 was applied, only 100 simulation steps were assumed because of a limited computer memory, so the corresponding multiple system of linear equations had 100 right hand sides and the total number of unknowns was $31\ 512 \times 100$, see Tab. 2.

4.4. Postprocessing

The aim of this example was to find a distribution of main principal stresses of elements in the structure. For computation of element stresses $Es = (\sigma_x, \sigma_y, \tau_{xy})$ from the element displacement vector we used the Calfem call 'planrs'. Then we calculated for each element in domain the maximum and the minimum values (variables denoted here as σ_1, σ_2) of the main principal stress in the following way:

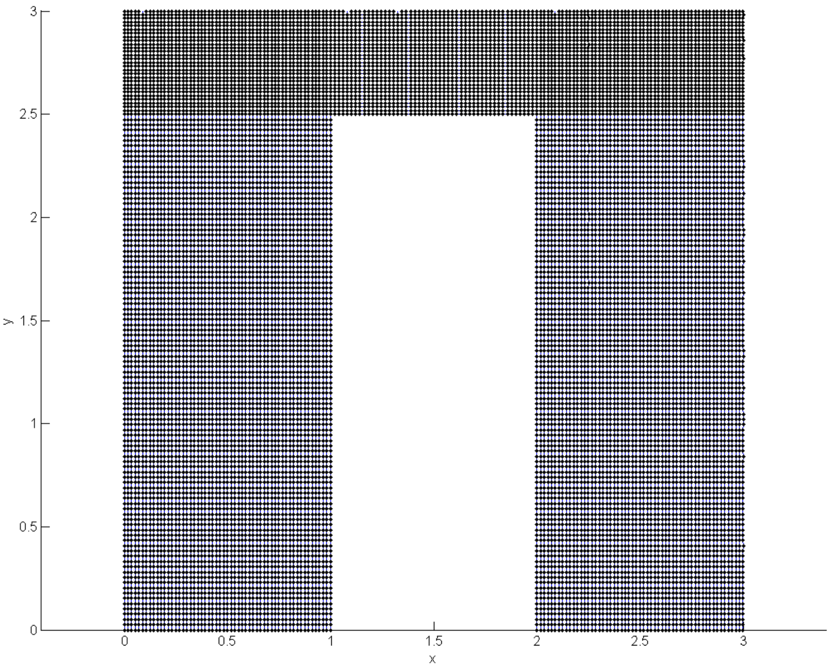

Figure 3: The finite element mesh of the frame using CALFEM (Problem name: FEM-Frame2_Geometry7).

Algorithm Myprincs [Compute principal stresses]

$s_p = 0.5(\sigma_x + \sigma_y + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2});$
$s_m = 0.5(\sigma_x + \sigma_y - \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2});$
$\sigma_1 = max(s_p, s_m);$
$\sigma_2 = min(s_p, s_m);$

Of course, the algorithm Myprinc was run in each simulation. These simulation results were subsequently statistically processed for Geometry5, see Tab. 4, Tab. 5 and for Geometry7 see Tab. 6. In order to obtain information about the distribution of the maximum value of the main principal stresses in geometry, the reliability analysis was computed in five subdomains of the structure separately. Tables contain results of the variable σ_1 taken from the algorithm Myprincs. For instance, the column denoted as $D5max$ contains probabilities of exceeding values of the first column of the table in the geometry domain no. 5.

The minimum observed value of σ_2 was $-1.25.10^6$ for Importance Sampling method with Geometry5. This observed value is very far from the critical value -20 MPa, so results of σ_2 were not printed.
Analysing results of Tab. 4 we can see that the direct Monte Carlo method did not detect extrem events in which the variable σ_1 was greater than $1.4.10^6$ at all. On the other hand, Importance Sampling method detected low probability cases $\sigma_1 > 2.10^6$.
The row of Tabs. 4 and 5 denotes as 'SF' contain results of the safety function 2. The probability of a failure was estimated by direct Monte Carlo method as $P_f = 1 - 0.911 = 0.0890$. When

Importance Sampling was applied, the probability of failure was estimated as $P_f = 1 - 0.961 = 0.0390$, .

When the finer FEM mesh Geometry7 was applied, the probability of failure was estimated as $P_f = 1 - 0.95184 = 0.04816$, see Tab. 6. This probability of failure was estimated only by 100 simulation steps because of limited computer memory.

Compared both analyses, the estimated probability of failures are not sensitive to mesh refinement. Let us accent that the Importance Sampling approach benefits the detection of low probability (critical) events. Of course, by repeating the probabilistic reliability assessment it is possible to obtain information about variance (i.e. accuracy) of estimated results.

Table 2: Properties of the multiple linear system of equations for the probabilistic reliability assessment example (Problem name: FEMFrame2_Geometry5_IS and FEMFrame2_Geometry7_IS, $reps = 0.0001$, $rdep = 10^{-6}$).

geometry	Geometry5	Geometry7
# unknowns per rhs	16 188	31 512
# rhs	1 000	100
# iterations	254	273
# mat-vec	595	739
Elapsed time (s)	546.578	208.438

Table 3: Comparison of the SBCG results for the probabilistic reliability assessment example.

coef	1.00E-08	1.00E-06	0.0001	0.1	1 (=SCG)
# iterations	308	254	267	747	446
# mat-vec	614	595	681	757	446
Master max.size	6	6	6	2	1
Elapsed time (s)	934.219	546.578	812.375	2316.48	1424.75

5. Conclusions and future work

In this paper, the Stable Block Conjugate Gradient Algorithm for the solution of linear systems with symmetric, positive definite matrices and with multiple right-hand sides is considered. The method was compared on a linear elasticity model arising from the probabilistic reliability assessment of a finite element model of a structure using the Monte Carlo method and Importance Sampling. The effect of mesh refinement was studied and discussed. In future work we would like to solve real 3D large large scale applications and dynamic reliability problems [1], [2]. The paper can be considered as a complement of the papers [6], and [20].

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Table 4: Probability to exceed selected values of the maximal principal stresses of each domains. Direct Monte Carlo results of the probabilistic reliability assessment (Geometry5, MC 1 000 steps).

Value	D1Max	D2Max	D3Max	D4Max	D5Max	GlobMax	SF
0	1	1	1	1	1	1	0.911
900000	0.001	0.152	0	0.002	0.045	0.152	0.005
1.E+6	0	0.073	0	0	0.013	0.073	0
1.1E+6	0	0.024	0	0	0.005	0.024	0
1.2E+6	0	0.008	0	0	0	0.008	0
1.3E+6	0	0.002	0	0	0	0.002	0
1.4E+6	0	0	0	0	0	0	0
1.5E+6	0	0	0	0	0	0	0
1.6E+6	0	0	0	0	0	0	0
1.7E+6	0	0	0	0	0	0	0
1.8E+6	0	0	0	0	0	0	0
1.9E+6	0	0	0	0	0	0	0
2.E+6	0	0	0	0	0	0	0

Table 5: Probability to exceed selected values of the maximal principal stresses of each domains. Importance Sampling results of the probabilistic reliability assessment (Geometry5, IS 1 000 steps).

Value	D1Max	D2Max	D3Max	D4Max	D5Max	GlobMax	SF
0	1	1	1	1	1	1	0.961
9e+5	0.00221	0.224	2.39e-7	0.00408	0.0137	0.226	0.00764
1e+6	0.00203	0.134	3.5e-11	0.00128	0.00739	0.136	0.000208
1.1e+6	6.02e-5	0.0129	0	8.09e-6	0.0041	0.013	1.25e-5
1.2e+6	2.46e-6	0.00452	0	2.98e-7	0.00145	0.00452	5.26e-6
1.3e+6	2.39e-7	0.00401	0	4.27e-9	0.000331	0.00401	0
1.4e+6	1.59e-9	0.00131	0	4.27e-9	5.76e-6	0.00131	0
1.5e+6	3.5e-11	4.08e-5	0	0	3.14e-7	4.08e-5	0
1.6e+6	0	2.67e-6	0	0	4.27e-9	2.67e-6	0
1.7e+6	0	2.45e-6	0	0	4.27e-9	2.45e-6	0
1.8e+6	0	4.42e-9	0	0	0	4.42e-9	0
1.9e+6	0	4.27e-9	0	0	0	4.27e-9	0
2e+6	0	4.27e-9	0	0	0	4.27e-9	0

Table 6: Probability to exceed selected values of the maximal principal stresses of each domains. Importance Sampling results of the probabilistic reliability assessment (Geometry7, IS 100 steps).

Value	D1Max	D2Max	D3Max	D4Max	D5Max	GlobMax	SF
0	1	1	1	1	1	1	0.95184
900000	0.00894	0.06116	8.64e-6	0.02238	0.05125	0.0644	0.00468
1e+6	0.00017	0.04793	0	0.00683	0.02261	0.04797	6.41e-5
1.1e+6	4.88e-5	0.02265	0	0.00480	0.01732	0.02273	5.45e-5
1.2e+6	4.88e-5	0.02238	0	1.15e-6	0.00505	0.02240	0
1.3e+6	3.61e-8	0.01806	0	5.76e-8	0.00485	0.01806	0
1.4e+6	0	0.01234	0	0	0.00067	0.01234	0
1.5e+6	0	0.00484	0	0	1.15e-6	0.00484	0
1.6e+6	0	0.00022	0	0	1.15e-6	0.00022	0
1.7e+6	0	1.15e-6	0	0	5.76e-8	1.15e-6	0
1.8e+6	0	5.76e-8	0	0	0	5.76e-8	0
1.9e+6	0	0	0	0	0	0	0
2e+6	0	0	0	0	0	0	0

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