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APPLICATION OF THE LOGARITHMIC STRAIN IN ACOUSTOELASTICITY

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Summary: *Expressions for velocities of elastic waves in pre-stressed solids are derived using the third order strain energy function and the Hencky's logarithmic strain tensor as the measure of deformation.*

1. Introduction

Acoustoelasticity describes how the velocity of the small amplitude sound waves propagating through an pre-stressed elastic medium is stress dependent. This phenomenon is a basis for non-destructive method to determine residual and active stresses in material and structure.

Hughes and Kelly [Hughes and Kelly 1953] introduced the theory and measured the effect of uniaxial stress on the velocity of sound waves in isotropic elastic material and have shown how the values of the three third-order elastic constants of an isotropic material can be determined. Later on, Thurston and Brugger [Thurston and Brugger 1964] extended the theory to the case of anisotropic material with arbitrary symmetry and Toupin and Bernstein [Toupin and Bernstein 1961] developed the theory further. However, all of them based their theories on the Green-Lagrange strain tensor and Murnaghan's strain energy function [Murnaghan 1951]. This strain energy function is polynomial of the third order. For isotropic material besides the ordinary Lamé constants λ and μ three third order elastic constants called Murnaghan's material parameters l, m and n must be used.

The frequently used Green-Lagrange strain tensor is easy and straightforward in its definition and application, but for the linear constitutive relations gives non-realistic material response, see e.g. [Batra 1998]. In acoustoelasticity, this property leads to the high sensitivity of the material parameters to the small errors in velocities measurements. Then, the sign of Murnaghan's parameter l changes for annealed and non-annealed aluminium alloy, [Kobayashi 1998]. Of course, all strain tensors are equal, but if the strain energy function is chosen as polynomial of the third order, the different strain tensors give different material response and the choice of the strain tensor is essential, see [Hoger 1999].

The material model based on the Hencky's logarithmic strain tensor was introduced and the wave velocities were analysed.

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2. Governing equation

The theory of acoustoelasticity superposes small dynamic deformations of an ultrasonic wave onto a static, finite deformation. For convenience, three configurations are introduced: initial, deformed and current configuration. The homogeneously pre-stressed configuration is called an initial configuration and we denote the values in initial configuration by the superscript I. Thus the coordinates of the material point in homogeneously deformed medium are $\mathbf{x}^I$.

The equation of motion referred to the initial configuration is

$$\frac{\partial \Pi_{ij}}{\partial X_j} = \rho^0 \frac{\partial^2 x_i}{\partial t^2} \tag{1}$$

as it is used in [Thurston and Brugger 1964].

We regard $\mathbf{\Pi}$ in (1) as a function of entropy and the deformation gradient $\mathbf{F}$. To obtain an appropriately linearized equation of motion, we expand $\mathbf{\Pi}$ in Taylor series about the initial state of coordinates $\mathbf{x}^I$

$$\Pi_{ij} - \Pi_{ij}^I = A_{ijmn} \left(\frac{\partial x_m}{\partial X_n} - \frac{\partial x_m^I}{\partial X_n} \right) + \dots \tag{2}$$

with

$$A_{ijmn} = \frac{\partial \Pi_{ij}}{\partial F_{mn}} \bigg|_{\substack{S=0 \\ \mathbf{x}=\mathbf{x}^I}} \tag{3}$$

The deviations from $\mathbf{x}^I$ to $\mathbf{x}$ are assumed to be explicitly isentropic and the tangent modulus $\mathbf{A}$ is expressed in the initial homogeneously pre-stressed configuration.

3. Solution of equation of motion

We suppose the solution of the wave equation (1)

$$u_k = u_k(k_m x_m^0 - \omega t) \tag{4}$$

where $\omega = 2\pi f$ is the angular velocity of the propagating wave, $\mathbf{k}$ is the wave vector,

$$k_m = n_m \frac{\omega}{c} \tag{5}$$

with $\mathbf{n}$ being the unit normal vector of the wave front and c the phase velocity. The phase velocity c is related to the original configuration.

Substituting the solution (4) into the equation of motion (1), we obtain

$$A_{ijkl} k_l k_j \delta_{jl} \frac{\partial^2 u_k}{\partial (k_m x_m^0 - \omega t)^2} = \rho_0 \omega^2 \frac{\partial^2 u_i}{\partial (k_m x_m^0 - \omega t)^2} \tag{6}$$

Which can be rewritten in the system of the homogenies equations

$$\left[A_{ijkl} k_l k_j - \rho_0 \omega^2 \delta_{ik} \right] \frac{\partial^2 u_k}{\partial (k_m x_m^0 - \omega t)^2} = 0 \qquad \text{for } \delta_{jl} \neq 0 \tag{7}$$

or

$$\left[A_{ijkl} n_l n_j - \rho_0 c^2 \delta_{ik} \right] \frac{\partial^2 u_k}{\partial (k_m x_m^0 - \omega t)^2} = 0 \qquad \text{for } \delta_{jl} \neq 0 \tag{8}$$

The acoustic tensor

$$\Delta_{ik} = A_{ijkl} n_l n_j \tag{9}$$

can be introduce. It is real and symmetric. If the acoustic tensor $\mathbf{\Delta}$ is positive definite, the equation has three real, positive eigenvectors and their eigenvectors are real and orthogonal.

The waves propagating in the direction $\mathbf{n} = \mathbf{e}_1$ are considered. Then the relation (8) can be rewritten as

$$\left[A_{i1k1} - \rho_0 c_0^2 \delta_{ik}\right] U_k = 0 \tag{10}$$

If the material is isotropic (or if the wave propagates in the directions of symmetry of an orthotropic material), $A_{2111} = A_{1121} = A_{3111} = A_{1131} = A_{2131} = A_{3121} = 0$. The system of equation (7) has diagonal coefficient matrix

$$\begin{bmatrix} A_{1111} - \rho_0 c_0^2 & 0 & 0 \\ 0 & A_{2121} - \rho_0 c_0^2 & 0 \\ 0 & 0 & A_{3131} - \rho_0 c_0^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

The necessary condition for the non-zero amplitudes $\mathbf{U}$ of the propagating waves is

$$\det \begin{pmatrix} A_{1111} - \rho_0 c_0^2 & 0 & 0 \\ 0 & A_{2121} - \rho_0 c_0^2 & 0 \\ 0 & 0 & A_{3131} - \rho_0 c_0^2 \end{pmatrix} = 0 \tag{12}$$

The equation (12) gives characteristic equation with three roots $A_{1111} - \rho_0 c_0^2$, $A_{2121} - \rho_0 c_0^2$ and $A_{3131} - \rho_0 c_0^2$. They correspond to the three eigenvectors $\mathbf{N}^{(1)} = \mathbf{e}_1$, $\mathbf{N}^{(2)} = \mathbf{e}_2$ and $\mathbf{N}^{(3)} = \mathbf{e}_3$.

The first eigenvalue $A_{1111} - \rho_0 \nu^2$ corresponds to the wave, which motion is parallel to the direction of propagation, i.e. in the direction $\mathbf{N}^{(1)} = \mathbf{e}_1$. The wave is called longitudinal (or pressure) wave and its velocity is

$$\rho_0 (c_{0\,\mathrm{L}})^2 = A_{1111} \tag{13}$$

The second and the third eigenvalues correspond to the plane waves, which motions are normal to the direction of propagation. They are called shear waves and its velocities are

$$\begin{aligned} \rho_0 (c_{0\,\mathrm{S1}})^2 &= A_{2121} \\ \rho_0 (c_{0\,\mathrm{S2}})^2 &= A_{3131} \end{aligned} \tag{14}$$

Both the longitudinal and shear waves expressed here are measured per undeformed matrix, it means in respect to the reference configuration.

4. Tangent modulus

The tangent modulus $\mathbf{A}$ (called also the first elasticity modulus, see [Marsden and Hughes 1983]) for the general strain tensor can be found in [Kruisová and Plešek 2005].

For investigation of small amplitude waves propagating in isotropic elastic media, the particular case for the diagonal deformation gradient

$$\mathbf{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{15}$$

can be used. For the Henckys's logarithmic strain tensor $\ln \mathbf{U}$

$$\ln \mathbf{U} = \begin{bmatrix} \ln \lambda_1 & 0 & 0 \\ 0 & \ln \lambda_2 & 0 \\ 0 & 0 & \ln \lambda_3 \end{bmatrix} \tag{16}$$

are components A_{1111}, A_{2121} and A_{3131} of the tangent modulus

$$\begin{aligned} A_{1111} &= -\frac{T_{11}}{\lambda_1^2} + \frac{H_{1111}}{\lambda_1^2} \\ A_{2121} &= \frac{T_{11} - T_{22}}{\lambda_1^2 - \lambda_2^2}\left[1 - 2\lambda_2^2 \frac{\ln \lambda_1 - \ln \lambda_2}{\lambda_1^2 - \lambda_2^2}\right] + \\ &\quad + \lambda_2^2 \frac{[\ln \lambda_1 - \ln \lambda_2]^2}{[\lambda_1^2 - \lambda_2^2]^2}\left(H_{1212} + H_{2112} + H_{2121} + H_{1221}\right) \\ A_{3131} &= \frac{T_{11} - T_{33}}{\lambda_1^2 - \lambda_3^2}\left[1 - 2\lambda_3^2 \frac{\ln \lambda_1 - \ln \lambda_3}{\lambda_1^2 - \lambda_3^2}\right] + \\ &\quad + \lambda_3^2 \frac{[\ln \lambda_1 - \ln \lambda_3]^2}{[\lambda_1^2 - \lambda_3^2]^2}\left(H_{1313} + H_{3113} + H_{3131} + H_{1331}\right) \end{aligned}$$

where $\mathbf{T}$ is the stress conjugate to the logarithmic strain tensor and $\mathbf{H}$ is the Hessian

$$H_{ijkl} = \frac{\partial^2 \psi}{\partial(\ln \mathbf{U})_{ij} \partial(\ln \mathbf{U})_{kl}} \tag{17}$$

For the second order constitutive relations the strain energy function of the third order is

$$\psi = \frac{1}{2}\lambda I_1^2 + 2\mu I_2 + \frac{1}{6}(2l - 2m + n)I_1^3 + (2m - n)I_1 I_2 + nI_3 \tag{18}$$

and the stress tensor $\mathbf{T}$ is

$$\begin{aligned} T_{ij} = \lambda I_1 \delta_{ij} &+ \frac{1}{2}\left(2l - 2m + n\right) I_1^2 \delta_{ij} + (2m - n) I_2 \delta_{ij} + \\ &+ 2\mu(\ln \mathbf{U})_{ij} + (2m - n) I_1 (\ln \mathbf{U})_{ij} + n(\ln \mathbf{U})_{im}(\ln \mathbf{U})_{mj} \end{aligned} \tag{19}$$

and components of Hessian of the strain energy function are

$$\frac{\partial^2 \psi}{\partial(\ln \mathbf{U})_{11}\, \partial(\ln \mathbf{U})_{11}} = \lambda + 2\mu + 2l I_1 + 4m(\ln \mathbf{U})_{11} \tag{20}$$

$$\frac{\partial^2 \psi}{\partial(\ln \mathbf{U})_{21}\, \partial(\ln \mathbf{U})_{21}} = 2\mu + (2\,m - n)\, I_1 + 2n(\ln \mathbf{U})_{21} \tag{21}$$

$$\frac{\partial^2 \psi}{\partial(\ln \mathbf{U})_{31}\, \partial(\ln \mathbf{U})_{31}} = 2\mu + (2\,m - n)\, I_1 + 2n(\ln \mathbf{U})_{31} \tag{22}$$

5. Results

Using the above equations, we obtain for the three different types of deformation these wave velocities

- hydrostatic pressure

$$\rho_0(c_{0\,\mathrm{L}})^2 = \Lambda + 2\mu - \frac{p}{3\Lambda+2\mu}\left(-5\Lambda - 6\mu + 6l + 4m\right)$$
$$\rho_0(c_{0\,\mathrm{S}})^2 = \mu - \frac{p}{3\Lambda+2\mu}\left(-2\mu + 3m - \frac{n}{2}\right)$$

- prestressed in longitudinal direction

$$\rho_0(c_{0\,\mathrm{L}})^2 = \Lambda + 2\mu + \left[6\Lambda + 6\mu + 4m + 2l + (3\Lambda + 4\mu + 4m)\frac{\Lambda}{\mu}\right]\frac{t}{3\Lambda+2\mu}$$
$$\rho_0(c_{0\,\mathrm{S}})^2 = \mu + \left[-\frac{\Lambda}{2} - \mu + m + \frac{n\Lambda}{4\mu}\right]\frac{t}{3\Lambda+2\mu}$$

- prestressed in transverse direction

$$\rho_0(c_{0\,\mathrm{L}})^2 = \Lambda + 2\mu + \left[2\Lambda + 4l + (\Lambda + 2l - 2m)\frac{\Lambda}{\mu}\right]\frac{t}{3\Lambda+2\mu}$$
$$\rho_0(c_{0\,\mathrm{S1}})^2 = \mu + \left[\Lambda + m - \frac{n}{2} - \frac{n\Lambda}{2\mu}\right]\frac{t}{3\Lambda+2\mu}$$
$$\rho_0(c_{0\,\mathrm{S2}})^2 = \mu + \left[-\frac{\Lambda}{2} - \mu + m - \frac{n\Lambda}{4\mu}\right]\frac{t}{3\Lambda+2\mu}$$

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