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SOLUTION OF THE BEAMS ON ELASTIC FOUNDATION (DETERMINISTIC AND PROBABILISTIC APPROACH)

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Summary: *The subject of this paper is the analysis of elastically supported beams (beams on elastic Winkler's foundation). The real beams of finite length can be also solved via superposition principle using the linear combinations of solutions of two beams of unlimited (infinite) length. The application of deterministic and probabilistic reliability assessment was used in a result evaluation (deflection, stress and buckling).*

1. Introduction

The analysis of bending of beams on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional at every point to the deflection of the beam at that point; etc., see reference Frydrýšek (2006) or Hetényi (1947).

The general problem of bending of beams on an elastic foundation is described by ordinary differential equation:

$$EJ_{ZT}\frac{d^4v}{dx^4}+\left(\frac{\beta kEJ_{ZT}}{GS}-N\right)\frac{d^2v}{dx^2}+kv=\mathrm{q}-\frac{dm}{dx}+\frac{\beta EJ_{ZT}}{GS}\frac{d^2\mathrm{q}}{dx^2}-\frac{\alpha_{\mathrm{t}}EJ_{ZT}}{\mathrm{h}}\frac{d^2(\mathrm{t}_2-\mathrm{t}_1)}{dx^2}+\cdots, \tag{1}$$

where EJ_{ZT} /Nm^2/ is bending stiffness of the beam, k /Pa/ is foundation stiffness, β /1/ is shear deflection constant of the beam, G /Pa/ is shear modulus of the beam, S /m^2/ is cross-sectional area of the beam and v /m/ is vertical deflection of the beam.

The beams can be loaded by: normal (axial) force N /N/, vertical force F /N/, intensity of vertical force q /Nm^{-1}/, moment M /N/, intensity of moment m /N/ and temperature gradient $\mathrm{t}_2-\mathrm{t}_1$ /$^{\mathrm{o}}$C/.

Consider that: $\beta=0$ (influence of shearing force is neglected), $m=0$ and $\mathrm{t}_2-\mathrm{t}_1=0$. Hence from (1) follows:

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$$EJ_{ZT}\frac{d^4v}{dx^4} - N\frac{d^2v}{dx^2} + kv = \mathrm{q} - \frac{dm}{dx}. \qquad (2)$$

The beams on elastic foundation can be classified into beams of infinite (or semi-infinite) length and beams of finite length L /m/, see fig.1.

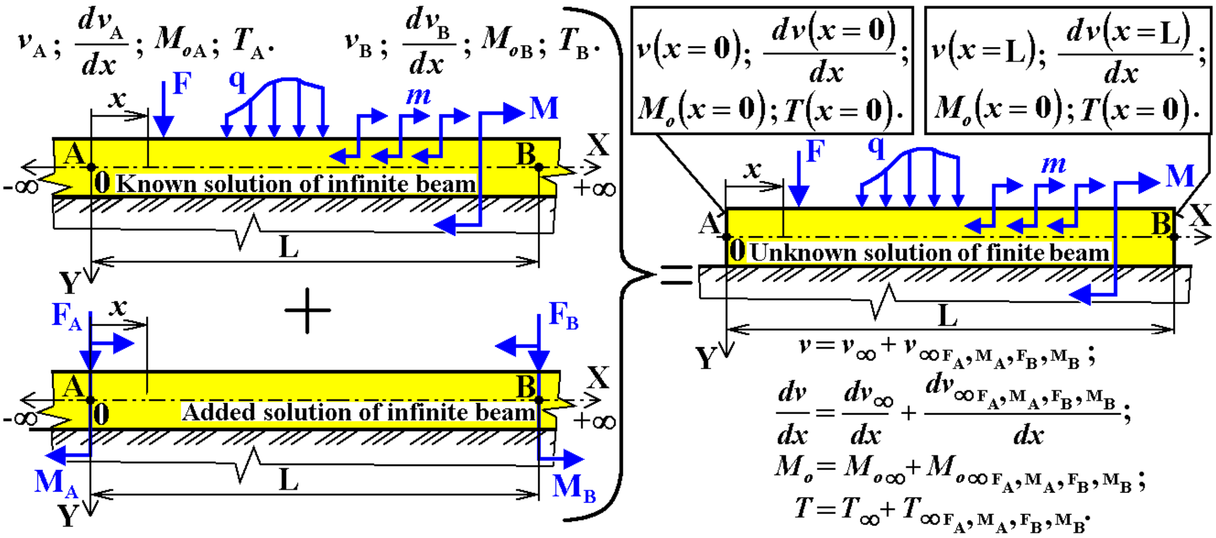
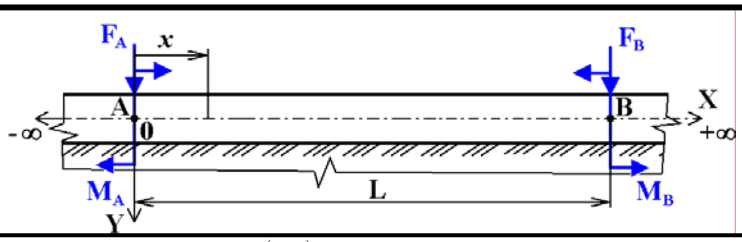


Fig.1. Superposition Principles Used for Solution of the General Beams of Finite Length L.

Assume that an infinite beam is subjected to a given loadings F, q, M and m as shown in the left top part of Fig.1. Because of this loadings certain values of boundary conditions i.e. deflections v_A, v_B /m/, slopes $\frac{dv_A}{dx}$, $\frac{dv_B}{dx}$ /rad/, bending moments M_{oA}, M_{oB} and shearing forces T_A, T_B will be produced at points A and B. By superposing on this loaded beam two pairs of concentrated forces and moments ($\mathrm{F_A}$, $\mathrm{M_A}$, $\mathrm{F_B}$ and $\mathrm{M_B}$, see left bottom part of Fig.1), we can modify the solution in such a way that at points A and B the required end conditions of finite beam will be fulfilled. These end-conditioning forces ($\mathrm{F_A}, \mathrm{F_B}$) and end-conditioning moments ($\mathrm{M_A}, \mathrm{M_B}$) are applied infinitely close to the outer side of the A-B portion. For each end (A and B) of the beam of finite length we can prescribe two boundary conditions.

Functions: v_∞ /m/, $\frac{dv_\infty}{dx}$ /rad/, $M_{o\infty}$ /Nm/ and T_∞ /N/ describe known solution of infinite beam i.e. deflection, slope, bending moment and shearing force. Functions: $v_{\infty \mathrm{F_A}, \mathrm{M_A}, \mathrm{F_B}, \mathrm{M_B}}$ /m/, $\frac{dv_{\infty \mathrm{F_A}, \mathrm{M_A}, \mathrm{F_B}, \mathrm{M_B}}}{dx}$ /rad/, $M_{o\infty \mathrm{F_A}, \mathrm{M_A}, \mathrm{F_B}, \mathrm{M_B}}$ /Nm/ and $T_{\infty \mathrm{F_A}, \mathrm{M_A}, \mathrm{F_B}, \mathrm{M_B}}$ /N/ describe added simple solution of infinite beam i.e. deflection, slope, bending moment and shearing force caused by concentrated forces and moments $\mathrm{F_A}$, $\mathrm{M_A}$, $\mathrm{F_B}$ and $\mathrm{M_B}$, see Tab.1 or more detail in reference Frydrýšek (2006).



$$\omega = \sqrt[4]{\frac{k}{4EJ_{\mathrm{ZT}}}}$$

$$v_{\infty\,\mathrm{F_A,M_A,F_B,M_B}} = \frac{\omega e^{\omega(x-\mathrm{L})}}{2k}\Big[\ \mathrm{F_B}\big(\cos[\omega(\mathrm{L}-x)]+\sin[\omega(\mathrm{L}-x)]\big)+2\omega\mathrm{M_B}\sin[\omega(\mathrm{L}-x)]\ \Big]+ + \frac{\omega e^{-\omega x}}{2k}\big[\mathrm{F_A}\left(\cos\omega x+\sin\omega x\right)+2\omega\mathrm{M_A}\sin\omega x\big]$$

$$\frac{dv_{\infty\,\mathrm{F_A,M_A,F_B,M_B}}}{dx} = \frac{\omega^2 e^{\omega(x-\mathrm{L})}}{k}\Big[\ \mathrm{F_B}\sin[\omega(\mathrm{L}-x)]+\omega\mathrm{M_B}\big(\sin[\omega(\mathrm{L}-x)]-\cos[\omega(\mathrm{L}-x)]\big)\ \Big]+ + \frac{\omega^2 e^{-\omega x}}{k}\big[\omega\mathrm{M_A}\left(\cos\omega x-\sin\omega x\right)-\mathrm{F_A}\sin\omega x\big]$$

$$M_{o\infty\,\mathrm{F_A,M_A,F_B,M_B}} = \frac{e^{-\omega x}}{4\omega}\big[\mathrm{F_A}\left(\cos\omega x-\sin\omega x\right)+2\omega\mathrm{M_A}\cos\omega x\big]+ + \frac{e^{\omega(x-\mathrm{L})}}{4\omega}\Big[\ \mathrm{F_B}\big(\cos[\omega(\mathrm{L}-x)]-\sin[\omega(\mathrm{L}-x)]\big)+2\omega\mathrm{M_B}\cos[\omega(\mathrm{L}-x)]\ \Big]$$

$$T_{\infty\,\mathrm{F_A,M_A,F_B,M_B}} = \frac{-e^{-\omega x}}{2}\big[\mathrm{F_A}\cos\omega x+\omega\mathrm{M_A}\left(\cos\omega x+\sin\omega x\right)\big]+ \frac{e^{\omega(x-\mathrm{L})}}{2}\Big[\ \mathrm{F_B}\big(\cos[\omega(\mathrm{L}-x)]\big)+\omega\mathrm{M_B}\big(\cos[\omega(\mathrm{L}-x)]+\sin[\omega(\mathrm{L}-x)]\big)\ \Big]$$

Tab.1. Solution of the Beam of Infinite Length (the Beam is Loaded by Forces $\mathrm{F_A}$, $\mathrm{F_B}$ and Moments $\mathrm{M_A}, \mathrm{M_B}$, Solution at the Interval $x \in (0;\mathrm{L})$, Derived for a Case when $N = 0$).

2. Example 1 - Clamped Beam (Deterministic Approach)

Consider an intensity of force: $\mathrm{q} = \mathrm{q_1}\dfrac{\mathrm{L}-x}{\mathrm{L}}$ distributed over length L of the beam with rectangular cross-section $\mathrm{b}\times\mathrm{h}$. The beam has clamped end A and free end B, see Fig.2. The task is to find solutions (deflection, slope, bending moment and shearing force) at the interval $x \in (0;\mathrm{L})$. Given: k; E; b; h; L; $\mathrm{q_1}$.

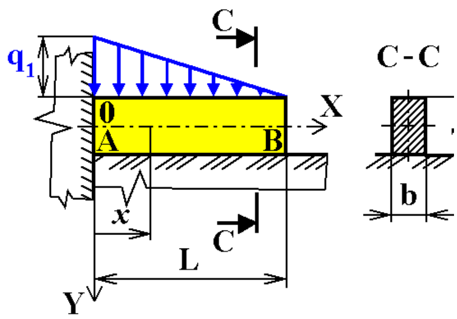


Fig.2. Solved Beam of Finite Lengtl

For the beam of length L loaded by triangle intensity of force (see Fig.2) are these boundary conditions:

$$v(x=0)=0\,,\quad \frac{dv(x=0)}{dx}=0\,,\quad M_o(x=\mathrm{L})=0 \ \text{ and } \ T(x=\mathrm{L})=0\,. \qquad (3)$$

Simple solution of infinite beam (caused by $\mathrm{F_A}$, $\mathrm{M_A}$, $\mathrm{F_B}$ and $\mathrm{M_B}$) is written in the Tab.1. Hence can be written:

$$v_{\infty\,F_A,M_A,F_B,M_B}(x=0)=\frac{\omega e^{-\omega L}}{2k}\left[\ F_B(\cos\omega L+\sin\omega L)+2\omega M_B\sin\omega L\ \right]+\frac{\omega F_A}{2k}\ , \qquad (4a)$$

$$\frac{dv_{\infty\,F_A,M_A,F_B,M_B}(x=0)}{dx}=\frac{\omega^2 e^{-\omega L}}{k}\left[\ F_B\sin\omega L+\omega M_B(\sin\omega L-\cos\omega L)\ \right]+\frac{\omega^3 M_A}{k}\ , \qquad (4b)$$

$$M_{o\infty\,F_A,M_A,F_B,M_B}(x=L)=$$
$$=\frac{e^{-\omega L}}{4\omega}\left[F_A(\cos\omega L-\sin\omega L)+2\omega M_A\cos\omega L\right]+\frac{F_B+2\omega M_B}{4\omega}\ , \qquad (4c)$$

$$T_{\infty\,F_A,M_A,F_B,M_B}(x=L)=$$
$$=\frac{-e^{-\omega L}}{2}\left[F_A\cos\omega L+\omega M_A(\cos\omega L+\sin\omega L)\right]+\frac{1}{2}\left[\ F_B+\omega M_B\ \right]. \qquad (4d)$$

The equations of equilibrium at point A:

$$v(x=0)=v_{\infty A}+v_{\infty\,F_A,M_A,F_B,M_B}(x=0),\qquad \frac{dv}{dx}(x=0)=\frac{dv_{\infty A}}{dx}+\frac{dv_{\infty\,F_A,M_A,F_B,M_B}(x=0)}{dx}.$$

According to (3) follow:

$$v_{\infty\,F_A,M_A,F_B,M_B}(x=0)=-v_{\infty A}\ ,\qquad \frac{dv_{\infty\,F_A,M_A,F_B,M_B}(x=0)}{dx}=-\frac{dv_{\infty A}}{dx}. \qquad (5a)$$

Analogous to (5a) can be written equations of equilibrium at point B:

$$M_{oB}=-M_{o\infty\,F_A,M_A,F_B,M_B}(x=L)\ ,\ \ T_B=-T_{\infty\,F_A,M_A,F_B,M_B}(x=L)\ . \qquad (5b)$$

Hence system of linear equations can be written with regard to equations (4) and (5):

$$\left[\ P_\infty\ \right]\times\left[\ F_A\quad M_A\quad F_B\quad M_B\ \right]^T=\left[\frac{-2k}{\omega}v_{\infty A}\quad \frac{-k}{\omega^3}\frac{dv_{\infty A}}{dx}\quad -2\omega M_{o\infty B}\quad 2T_{\infty B}\right]^T,$$

where:

$$\left[\ P_\infty\ \right]=\begin{bmatrix} 1 & 0 & \dfrac{\cos\omega L+\sin\omega L}{e^{\omega L}} & \dfrac{2\omega\sin\omega L}{e^{\omega L}} \\ 0 & 1 & \dfrac{\sin\omega L}{\omega e^{\omega L}} & \dfrac{\sin\omega L-\cos\omega L}{e^{\omega L}} \\ \dfrac{\cos\omega L-\sin\omega L}{2e^{\omega L}} & \dfrac{\omega\cos\omega L}{e^{\omega L}} & \dfrac{1}{2} & \omega \\ \dfrac{\cos\omega L}{e^{\omega L}} & \dfrac{\omega(\cos\omega L+\sin\omega L)}{e^{\omega L}} & -1 & -\omega \end{bmatrix}.$$

This leads to the solution of system of linear equations which is:

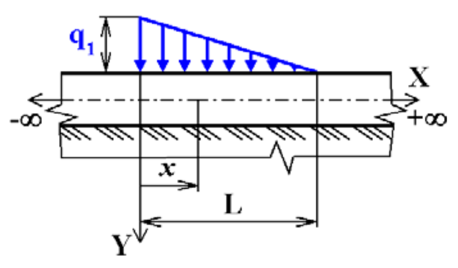
$$\begin{bmatrix} F_A & M_A & F_B & M_B \end{bmatrix}^T = \begin{bmatrix} P_\infty \end{bmatrix}^{-1} \times \begin{bmatrix} \dfrac{-2k}{\omega} v_{\infty A} & \dfrac{-k}{\omega^3}\dfrac{dv_{\infty A}}{dx} & -2\omega M_{o\infty B} & 2T_{\infty B} \end{bmatrix}^T , \tag{6}$$

where:

$$\begin{bmatrix} P_\infty \end{bmatrix}^{-1} = \begin{bmatrix} P_{INV\infty_{1,1}} & P_{INV\infty_{1,2}} & P_{INV\infty_{1,3}} & P_{INV\infty_{1,4}} \\ P_{INV\infty_{2,1}} & P_{INV\infty_{2,2}} & P_{INV\infty_{2,3}} & P_{INV\infty_{2,4}} \\ P_{INV\infty_{3,1}} & P_{INV\infty_{3,2}} & P_{INV\infty_{3,3}} & P_{INV\infty_{3,4}} \\ P_{INV\infty_{4,1}} & P_{INV\infty_{4,2}} & P_{INV\infty_{4,3}} & P_{INV\infty_{4,4}} \end{bmatrix} . \tag{7}$$

Matrix $\begin{bmatrix} P_\infty \end{bmatrix}^{-1}$ is inverse square matrix which is composed of 16 elements. For more information about $P_{INV\infty_{i,j}}$ elements see reference Frydrýšek (2005) or Frydrýšek (2006).

Solution of v_∞, $\frac{dv_\infty}{dx}$, $M_{o\infty}$ and T_∞ is known solution of infinite beam which is derived for the intervals $x \in \langle 0; L)$ in reference Hetényi (1947) and Frydrýšek (2006), see also Tab.2.

 	$\omega = \sqrt[4]{\dfrac{k}{4EJ_{ZT}}}$	$q = \dfrac{q_1}{L}(L - x)$	$x \in \langle 0; L)$
	$v_\infty = \dfrac{q_1}{4\omega k L}\Big[e^{-\omega x}\left[\sin\omega x - (1+2\omega L)\cos\omega x\right] + e^{\omega(x-L)}\left(\cos[\omega(L-x)] - \sin[\omega(L-x)]\right) + 4\omega(L-x)\Big]$		
$\dfrac{dv_\infty}{dx} = \dfrac{q_1}{2kL}\Big[2 - e^{\omega(x-L)}\cos[\omega(L-x)] - e^{-\omega x}\left([1+\omega L]\cos\omega x + \omega L \sin\omega x\right)\Big]$			
$M_{o\infty} = \dfrac{q_1}{8L\omega^3}\Big[e^{-\omega x}(\cos\omega x + [1+2\omega L]\sin\omega x) - e^{\omega(x-L)}\left(\cos[\omega(L-x)] + \sin[\omega(L-x)]\right)\Big]$			
$T_\infty = \dfrac{q_1}{4L\omega^2}\Big[e^{-\omega(L-x)}\sin[\omega(L-x)] + e^{-\omega x}\left([1+\omega L]\sin\omega x - \omega L\cos\omega x\right)\Big]$			

Tab.2 Known Solution of Infinite Beam at the Interval $x \in \langle 0; L)$.

Hence from the Tab.2 follow:

$$v_{\infty A} = v_\infty(x=0) = \frac{q_1}{4\omega k L}\left[e^{-\omega L}\left(\cos\omega L - \sin\omega L\right) + 2\omega L - 1\right] , \tag{8a}$$

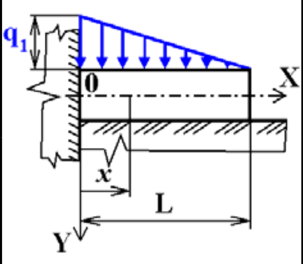
$$\frac{dv_{\infty A}}{dx} = \frac{dv_\infty(x=0)}{dx} = \frac{q_1}{2kL}\left[1 - \omega L - e^{-\omega L}\cos\omega L\right] , \tag{8b}$$

$$M_{o\infty B} = M_{o\infty}(x=L) = \frac{q_1}{8L\omega^3}\left(e^{-\omega L}[\cos\omega L + (1+2\omega L)\sin\omega L] - 1\right) , \tag{8c}$$

$$T_{\infty B} = T_\infty(x = L) = \frac{q_1}{4L\omega^2} e^{-\omega L}\left[(1+\omega L)\sin\omega L - \omega L\cos\omega L\right] . \qquad (8d)$$

The end-conditioning forces and moments (6) with using of (8) are given by:

$$\begin{Bmatrix} F_A \\ M_A \\ F_B \\ M_B \end{Bmatrix} = \left[\ P_\infty\ \right]^{-1} \times \begin{Bmatrix} \frac{-2k}{\omega} v_{\infty A} \\ \frac{-k}{\omega^3}\frac{dv_{\infty A}}{dx} \\ -2\omega M_{oB} \\ 2T_B \end{Bmatrix} = \frac{q_1}{2\omega^2 L}\left[\ P_\infty\ \right]^{-1} \times \begin{Bmatrix} e^{-\omega L}(\sin\omega L - \cos\omega L) - 2\omega L + 1 \\ \frac{e^{-\omega L}\cos\omega L + \omega L - 1}{\omega} \\ \frac{1 - e^{-\omega L}\left[\cos\omega L + (1+2\omega L)\sin\omega L\right]}{2} \\ e^{-\omega L}\left[(1+\omega L)\sin\omega L - \omega L\cos\omega L\right] \end{Bmatrix} . \qquad (9)$$

 	$\overline{B}_1 = \frac{2q_1(\cos\omega L + \omega L\sin\omega L)}{k\omega L(\cos 2\omega L + \cosh 2\omega L + 2)}$ $\overline{B}_3 = \frac{2q_1(\cosh\omega L - \omega L\sinh\omega L)}{k\omega L(\cos 2\omega L + \cosh 2\omega L + 2)}$	$\overline{B}_2 = \frac{2q_1\cos\omega L}{k(\cos 2\omega L + \cosh 2\omega L + 2)}$ $\overline{B}_4 = \frac{2q_1\cosh\omega L}{k(\cos 2\omega L + \cosh 2\omega L + 2)}$
$v = \frac{q_1(L-x)}{kL} + \left(\overline{B}_1\sinh\omega x - \overline{B}_2\cosh\omega x\right)\cos[\omega(L-x)] + \left(\overline{B}_3\sin\omega x - \overline{B}_4\cos\omega x\right)\cosh[\omega(L-x)]$		$\omega = \sqrt[4]{\frac{k}{4EJ_{ZT}}}$ $q = \frac{q_1}{L}(L-x)$
$\frac{dv}{dx} = \omega\Big[\left(\overline{B}_1\cosh\omega x - \overline{B}_2\sinh\omega x\right)\cos[\omega(L-x)] + \left(\overline{B}_1\sinh\omega x - \overline{B}_2\cosh\omega x\right)\sin[\omega(L-x)] + \left(\overline{B}_3\cos\omega x + \overline{B}_4\sin\omega x\right)\cosh[\omega(L-x)] + \left(\overline{B}_4\cos\omega x - \overline{B}_3\sin\omega x\right)\sinh[\omega(L-x)]\Big] - \frac{q_1}{kL}$		
$M_o = \frac{k}{2\omega^2}\Big[\left(\overline{B}_2\sinh\omega x - \overline{B}_1\cosh\omega x\right)\sin[\omega(L-x)] + \left(\overline{B}_3\cos\omega x + \overline{B}_4\sin\omega x\right)\sinh[\omega(L-x)]\Big]$		
$T = \frac{k}{2\omega}\Big[\left(\overline{B}_2\cosh\omega x - \overline{B}_1\sinh\omega x\right)\sin[\omega(L-x)] + \left(\overline{B}_1\cosh\omega x - \overline{B}_2\sinh\omega x\right)\cos[\omega(L-x)] + \left(\overline{B}_4\cos\omega x - \overline{B}_3\sin\omega x\right)\sinh[\omega(L-x)] - \left(\overline{B}_3\cos\omega x + \overline{B}_4\sin\omega x\right)\cosh[\omega(L-x)]\Big]$		

Tab.3 Fouded Solution of the Finite Beam at the Interval $x \in (0; L)$.

For chosen values: $b = b_{DET} = 0.026\,m$; $h = h_{DET} = 0.05\,m$; $L = L_{DET} = 1.097\,m$; $E = E_{DET} = 2.079\times10^{11}$ Pa; $K = K_{DET} = 3\times10^7$ Nm^{-3}; $q_1 = q_{1\,DET} = 10^4$ Nm^{-1} (see Fig.2) can be written:

$$k_{DET} = K_{DET} b_{DET} = 3\times10^{7}\times0.026 = 7.8\times10^{5}\ \text{Pa}, \tag{10}$$

$$\omega_{DET} = \sqrt[4]{\frac{k_{DET}}{4E_{DET} J_{ZT\,DET}}} = \sqrt[4]{\frac{3K_{DET}}{E_{DET} h^{3}_{DET}}} = \sqrt[4]{\frac{3\times3\times10^{7}}{2.079\times10^{11}\times0.05^{3}}} = 1.364173\ \text{m}^{-1}. \tag{11}$$

Lower subscript “DET” means deterministic approach.

Hence from the Tab.3 follow: $\overline{B}_1 = \overline{B}_{1\,DET} = 2.438253\times10^{-3}$ m, $\overline{B}_2 = \overline{B}_{2\,DET} = 1.728926\times10^{-4}$ m, $\overline{B}_3 = \overline{B}_{3\,DET} = -1.290544\times10^{-3}$ m, $\overline{B}_4 = \overline{B}_{4\,DET} = 5.461773\times10^{-3}$ m.

The values of deflection, slope, bending moment and shearing force over length $L_{DET} = 1.097$ m are shown on Fig.3 and 4.

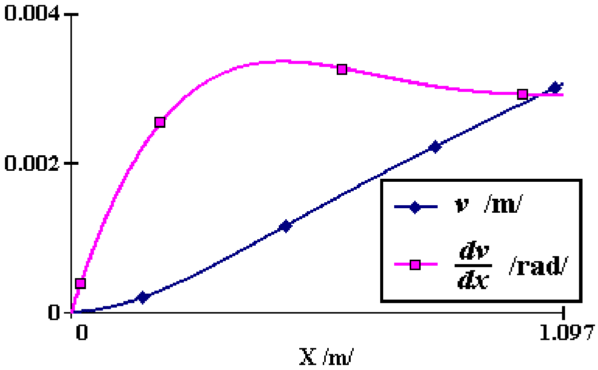


Fig.3. Deflection and Slope Diagrams (Deterministic Approach).

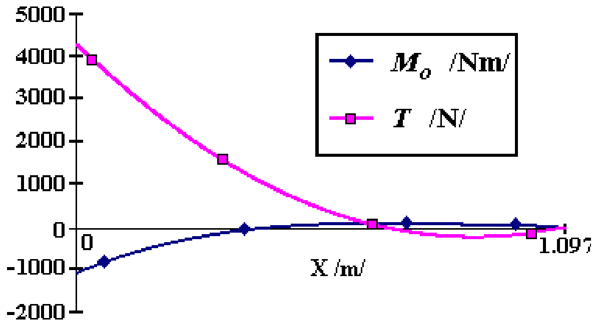


Fig.4. Bending Moment and Shearing Force Diagrams (Deterministic Approach).

From the Fig.3 and 4 and Tab.3 is evident that maximal values of deflection, bending moment and shearing force are:

$$v_{MAX\,DET} = v(x = L_{DET}) =$$
$$= \overline{B}_1 \sinh\omega_{DET} L_{DET} - \overline{B}_2 \cosh\omega_{DET} L_{DET} + \overline{B}_3 \sin\omega_{DET} L_{DET} - \overline{B}_4 \cos\omega_{DET} L_{DET} =$$
$$= 3.073827\times10^{-3}\ \text{m}; \tag{12}$$

$$M_{o\ MAX\,DET} = \left|M_o(x=0)\right| = \frac{k_{DET}}{2\omega^2_{DET}} \left|\overline{B}_3 \sinh\omega_{DET} L_{DET} - \overline{B}_1 \sin\omega_{DET} L_{DET}\right|$$
$$= 1.083225\times10^{3}\ \text{Nm}; \tag{13}$$

$$T_{MAX\,DET} = T(x=0) = \frac{k_{DET}}{2\omega_{DET}} \times$$
$$\times\left(\overline{B}_1 \cos\omega_{DET} L_{DET} + \overline{B}_2 \sin\omega_{DET} L_{DET} - \overline{B}_3 \cosh\omega_{DET} L_{DET} + \overline{B}_4 \sinh\omega_{DET} L_{DET}\right) =$$
$$= 4.278137\times10^{3}\ \text{N}. \tag{14}$$

For the maximal values of bending stress $\sigma_{\text{MAX DET}}$ and shearing stress $\tau_{\text{MAX DET}}$ acquired via deterministic approach can be written:

$$\sigma_{\text{MAX DET}} = \frac{6M_{o\ \text{MAX}_{\text{DET}}}}{\text{b}_{\text{DET}}\text{h}^2_{\text{DET}}} = 99.99\times10^6\ \text{Pa} = 99.99\ \text{MPa}\ , \tag{15}$$

$$\tau_{\text{MAX DET}} = \frac{3T_{\text{MAX}_{\text{DET}}}}{2\text{b}_{\text{DET}}\text{h}_{\text{DET}}} = 4.936311\times10^6\ \text{Pa} = 4.94\ \text{MPa}\ . \tag{16}$$

3. Example 1 - clamped beam (probabilistic approach)

Compare maximal deterministic deflection and stresses with probabilistic deflection and stresses of the beam on the fig.2. Beam is subjected to dead load with maximal value $\text{q}_1 = 10^4\ \text{Nm}^{-1}$.

Chosen nominal values: ($\text{b}_{\text{DET}} = 0.026\ \text{m}$; $\text{h}_{\text{DET}} = 0.05\ \text{m}$; $\text{L}_{\text{DET}} = 1.097\ \text{m}$; $E_{\text{DET}} = 2.079\times10^{11}\ \text{Pa}$; $K_{\text{DET}} = 3\times10^7\ \text{Nm}^{-3}$) and maximal value of dead loading $\text{q}_{1\,\text{DET}} = 10^4\ \text{Nm}^{-1}$ are the same like in the former chapter 2.

Variation of beam modulus of elasticity E is represented by histogram Evar: N1−15.his. Variation of width b and high h of the beam are expressed by histograms bvar, hvar: N1−03.his. Variation of span L of the beam is represented by histogram Lvar: N1−01.his. Variation of modulus of foundation K is expressed by histogram Kvar: N1−30.his. Variation of loading q_1 is represented by histogram q1var: DEAD1.his. For more detail about presented histograms see reference Marek (2001).

Hence the input probabilistic variables are: $E_{\text{PRO}} = E_{\text{DET}}\times\text{Evar}$; $\text{b}_{\text{PRO}} = \text{b}_{\text{DET}}\times\text{bvar}$; $\text{h}_{\text{PRO}} = \text{h}_{\text{DET}}\times\text{hvar}$; $\text{L}_{\text{PRO}} = \text{L}_{\text{DET}}\times\text{Lvar}$; $K_{\text{PRO}} = K_{\text{DET}}\times\text{Kvar}$; $\text{q}_{1\,\text{PRO}} = \text{q}_{1\,\text{DET}}\times\text{q1var}$; $k_{\text{PRO}} = K_{\text{PRO}}\times\text{b}_{\text{PRO}}$.

Lower subscript “PRO” means probabilistic approach.

Using the Anthill programme (SBRA method using Monte Carlo simulation, see Marek 2001) and modified formulas from the Tab.3 or modified equations (11) to (16), can be calculated probabilistic values: $M_{o\,\text{MAX PRO}}$, $T_{\text{MAX PRO}}$, $\sigma_{\text{MAX PRO}}$, $\tau_{\text{MAX PRO}}$ and $v_{\text{MAX PRO}}$. These probabilistic values are compared with the deterministic values: k_{DET}, $\text{q}_{1\,\text{DET}}$, $M_{o\,\text{MAX DET}}$, $T_{\text{MAX DET}}$, $\sigma_{\text{MAX DET}}$, $\tau_{\text{MAX DET}}$ and $v_{\text{MAX DET}}$, see for example fig.5 to 8.

From the fig.5 is evident that $\text{q}_{1\,\text{DET}}$ is also maximum of intensity of force.

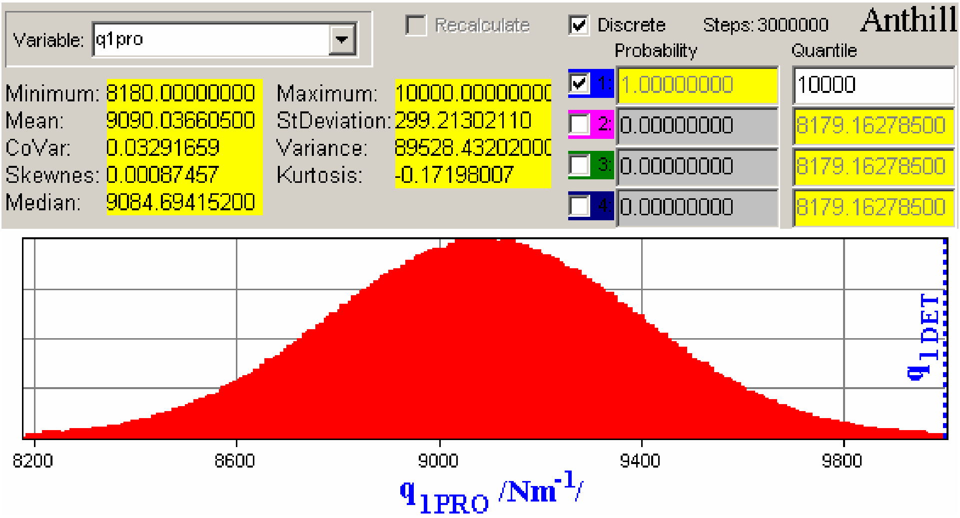


Fig.5 Comparing of Probabilistic and Deterministic Approach (Values of Magnitude of Intensity of Force).

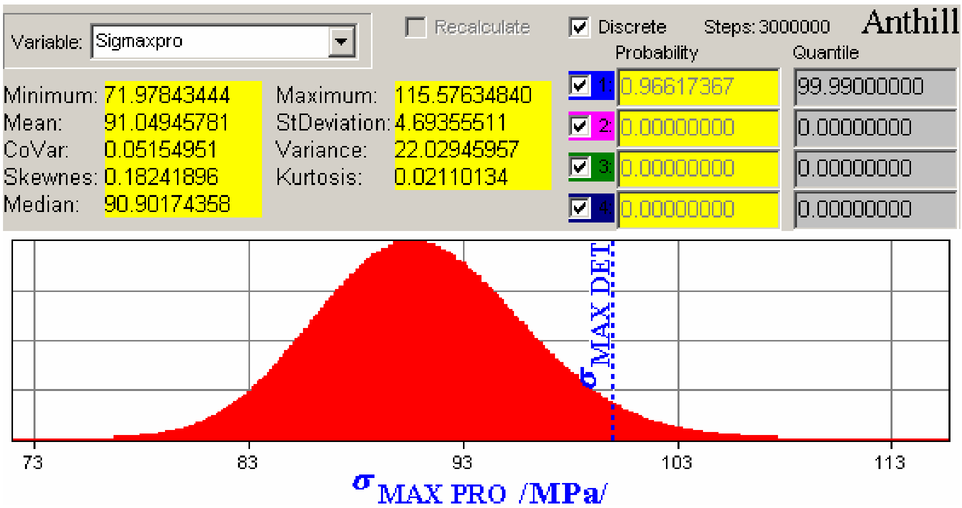


Fig.6 Comparing of Probabilistic and Deterministic Approach (maximal values of bending stresses).

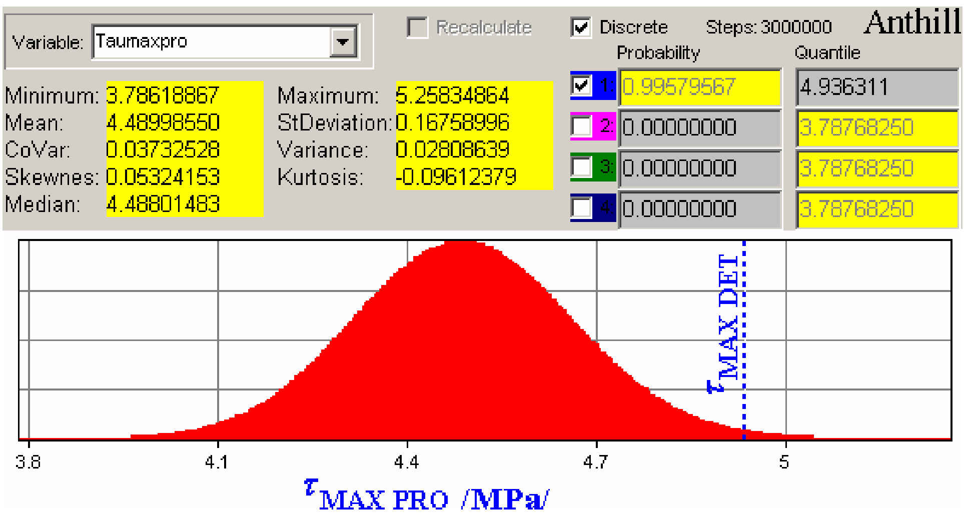


Fig.7 Comparing of Probabilistic and Deterministic Approach (Maximal Values of Shearing Stresses).

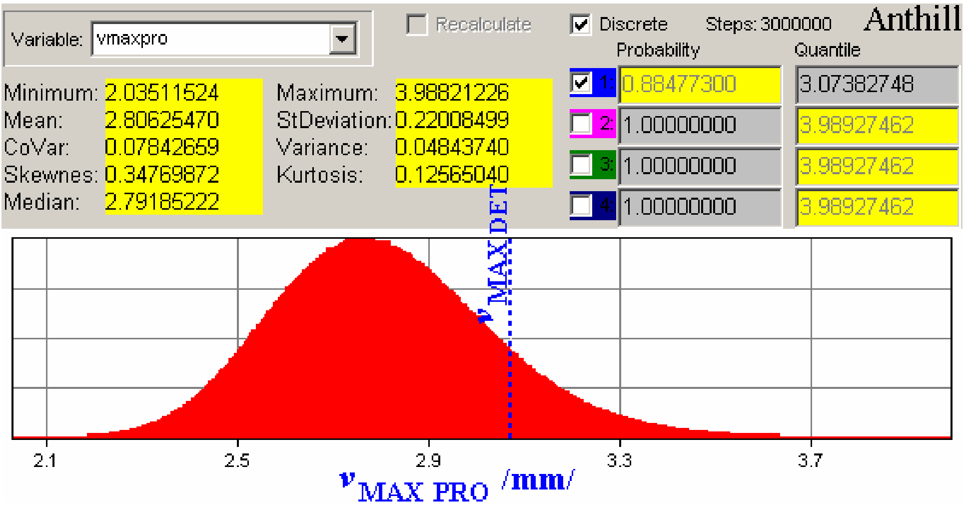


Fig.8 Comparing of Probabilistic and Deterministic Approach (Maximal Values of Deflection).

Some results of deterministic and probabilistic approaches are compared in the tab.4, where Operator “ P ” means probability.

Deterministic approach:	**Probabilistic approach:**			**Probability (Comparing):**
	Minimum value:	**Mean value:**	**Maximum value:**	
$q_{1\,DET} = 10^4\ \mathrm{Nm}^{-1}$	$q_{1\,PRO}$:			$P(q_{1\,PRO} \leq q_{1\,DET}) = 1$
	8180 Nm-1	9090.04 Nm-1	104 Nm-1	
$k_{DET} = 0.78$ MPa	k_{PRO} :			$P(k_{DET} < k_{PRO} = 0.5)$
	0.53 MPa	0.78 MPa	1.04 MPa	
$\sigma_{MAX\,DET} = 99.99$ MPa	$\sigma_{MAX\,PRO}$:			$P(\sigma_{MAX\,DET} < \sigma_{MAX\,PRO}) = 0.96$
	71.98 MPa	91.05 MPa	115.58 MPa	
$\tau_{MAX\,DET} = 4.94$ MPa	$\tau_{MAX\,PRO}$:			$P(\tau_{MAX\,DET} < \tau_{MAX\,PRO}) = 0.996$
	3.79 MPa	4.49 MPa	5.26 MPa	
$v_{MAX\,DET} = 3.07\ mm$	$v_{MAX\,PRO}$:			$P(v_{MAX\,DET} < v_{MAX\,PRO}) = 0.885$
	2.04 mm	2.81 mm	3.99 mm	

Tab.4 Some Deterministic and Probabilistic Results and their Comparing.

4. Example 2 - Buckling of General Infinite Beam (Probabilistic Approach)

For buckling of general infinite beams on elastic foundation loaded also by an axial negative force $N < 0$ (see eq.(2)), can be derived formula for critical value of normal force:

$$|N_{CR}| = 2\sqrt{kEJ_{ZT}}\ \ /\mathrm{N}/\,. \qquad (15)$$

Using the Anthill programme, can be calculated probabilistic values:

$$\left| N_{\mathrm{CR\ PRO}} \right| = 2\sqrt{k_{\mathrm{PRO}} E_{\mathrm{PRO}} J_{\mathrm{ZT\ PRO}}}\ , \qquad (16)$$

see fig.9 and for more details reference Frydrýšek (2006).

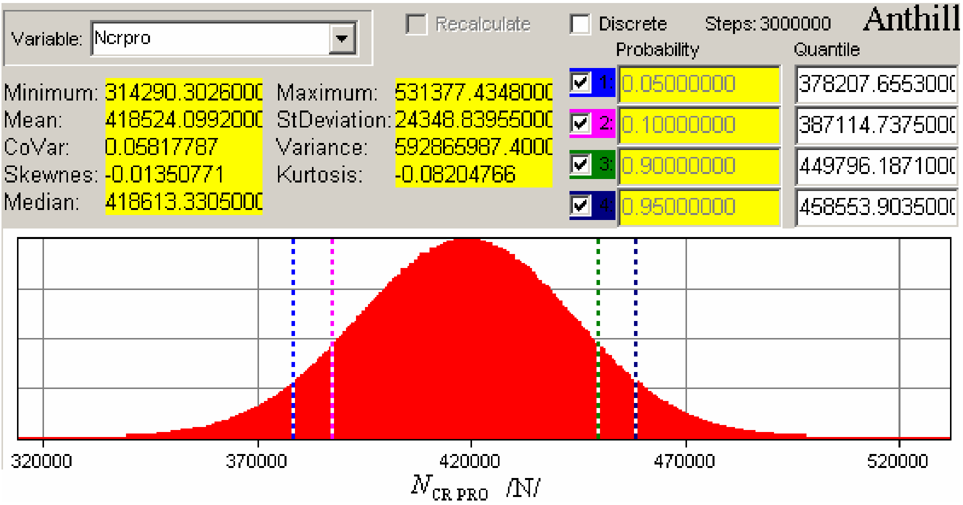


Fig.9 Critical Values of Axial (Buckling) Force for an Infinite Beam on Elastic Foundation.

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