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WAVELET DECOMPOSITION OF NON-STATIONARY RANDOM PROCESSES

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Summary: *The paper provides a support for the wavelet based approach for the analysis of the linear MDOF systems subjected to non-stationary excitation, as was proposed by [Basu and Gupta 1997]. It tries to cover the lack of input data in the form of the wavelet decomposition of seismic records. Examples of the wavelet decomposition of the true European accelerograms are provided and the simplified description of the wavelet spectrum is proposed.*

1. Introduction

The study of the dynamic behaviour of linear and non-linear systems to random excitation is of great importance in reliability and safety analysis in engineering practice. Recently, there has been an increasing interest in direct stochastic integration schemes due to their simplicity and applicability to multi-degree-of-freedom linear and non-linear systems, see e.g. [To 1992, Náprstek et al. 1995]

Simulation of effects of the ground motion on structures is usually performed using two approaches: either deterministic, or stochastic. The stochastic approach requires certain stochastic description of the input motion, which has strong connection with a certain seismic event. Among the stochastic characteristics, which are used in stochastic mechanics, usually belong the spectral content (possibly evolutionary) represented by the power spectra, response spectra, deterministic envelope, assumption of the Gaussian character of the random part of vibration. Moreover, it is usual to assume a certain shape of the power spectral density (e.g. Kanai-Taimi PSD of other ARMA model) or the shape of the deterministic envelope or modulation (e.g. exponential or the famous Saragoni & Hart modulation function).

Random processes, originating from the natural sources – like earthquakes or wind – are characterized by a high non-stationarity and, in most cases, by a marked variability of frequency contents. Description of temporal variation of the amplitude is usually adopted in the mathematical model of the behaviour of the structure: the general input motion is usually modeled as an uniformly modulated stationary process [Bolotin 1961, Saragoni and Hart 1974].

$$v(t) = m(t)v_{\mathrm{stat}}(t) \tag{1}$$

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The individual approaches distinct in the choice of the modulation function, varying form the Unit Step (Heaviside) function [Housner 1947] or exponentially raising and/or decaying function [Bolotin 1961, Saragoni and Hart 1974] up to piecewise linear or quadratic function [Fischer 2002].

There has been developed various explicit exact or approximative formulas for estimation of the response of the classically or non-classically damped structures, subjected to the temporally modulated stationary process. These formulas usually heavily depend on the the model chosen for the input motion. However, some models allow certain modification for a more general formulation of the input motion.

The non-stationarity in the frequency content can be approximated by several other approaches. Among them one can include usage of the Evolutionary power spectra [Priestley 1965], Windowed Fourier transform, Empirical mode decomposition [Huang 1996] or Wavelet transform [Basu and Gupta 1997, Lin and Yong 1987]. Sometimes it is possible to formulate the description of the fully non-stationary input motion using terms suitable for the classical formulas [To 1992, Náprstek, et al. 1993, Náprstek et al. 1995]. E.g. the wavelet based Multi-resolution analysis [Malat 1989], which leads to the estimation of the Evolutionary power spectra, can be formulated as the sum of temporally modulated stationary processes, which are statistically orthogonal [Fischer 2002].

To use the full power of modern tools of non-stationary signal analysis it is necessary to develop the brand new approaches. Basu and Gupta (1997) have proposed a new wavelet-based framework without the assumptions of a specific modulating function. For the analysis of the linear MDOF systems subjected to non-stationary excitation they obtain the closed-form explicit solutions for the moments of the instantaneous power spectral density in a generalized form. They use the wavelet basis close to both Littlewood-Paley basis and Harmonic wavelet basis, as it is described in [Newland 1993].

The mentioned approach seems to be promising, but — as the authors admit — it lacks sufficient input data. It is not very common practice to describe the random vibration using the wavelet coefficients.

In opposite to the simple case, where the non-stationary random process is fully described by a few coefficients of the ARMA stationary process and the modulation function, the wavelet coefficients span in the limiting case a rectangle in the time-frequency plane. To simply describe such a structure we propose to use functions approximating values of wavelet coefficients.

2. Wavelet transform

The ground motion is considered as a zero-mean, non-stationary process $z(t)$ with locally Gaussian characteristics. The Wavelet transform $W_\psi z(a,b)$ is defined as follows (see e.g. [Newland 1993])

$$W_\psi(a,b) = \int_{-\infty}^{\infty} z(t)\psi_{a,b}(t)\mathsf{d}t, \qquad a > 0 \tag{2}$$

where the translated and dilated form $\psi_{a,b}(t)$ of the real wavelet basis $\psi(t)$ is defined by

$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right) \tag{3}$$

The inverse wavelet transform is defined by

$$z(t) = \frac{1}{2\pi C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^2} W_\psi z(a,b) \psi_{a,b}(t) \mathrm{d}a \mathrm{d}b \tag{4}$$

where

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \mathrm{d}\omega < \infty \tag{5}$$

In equation (5), the symbol $\hat{\psi}(\omega)$ stands for the Fourier transformation of the mother wavelet $\psi(t)$, which is in this case defined as

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(t) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}t \tag{6}$$

Translation and dilatation of the mother wavelet, as it is used in the equation (3), is controlled by the parameters a and b: value of b centers the basis function in the neighborhood of $t = b$ and the parameter a defines the frequency scale as can be seen from the Fourier transform of the translated and dilated basis function:

$$\widehat{\psi_{a,b}}(\omega) = \hat{\psi}(a\omega) \mathrm{e}^{\mathrm{i}\omega b} \tag{7}$$

Denoting the center and the radius of the mother wavelet

$$t^\star = \frac{1}{||\psi(t)||_2^2} \int_{-\infty}^{\infty} t |\psi(t)|^2 \mathrm{d}t \tag{8}$$

$$\Delta_\psi = \frac{1}{||\psi(t)||_2} \left(\int_{-\infty}^{\infty} (t - t^\star)^2 |\psi(t)|^2 \mathrm{d}t \right)^{1/2} \tag{9}$$

one can easily see that the center and radius of the translated and dilated basis function $\psi_{a,b}(t)$ is $b + at^\star$ and $a\Delta_\psi$ respectively.

3. Wavelet coefficients

Basu and Gupta use the slightly modified L-P basis function, which is characterized by a reasonably fast temporal decay and excellent frequency localization, as the Fourier transform of this basis has compact support. Namely,

$$\begin{aligned} \psi(t) &= \frac{1}{\pi\sqrt{\sigma - 1}} \frac{\sin \sigma\pi t - \sin \pi t}{t}, \\ \hat{\psi}(\omega) &= \begin{cases} (2(\sigma - 1)\pi)^{-1/2} & \pi \leq |\omega| \leq \sigma\pi \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{10}$$

Moreover, the translated and dilated form of this wavelet leads to a family of mutually orthogonal functions. This helps in simplifying the input-output relationship of a linear dynamical system. Since the frequency bands corresponding to this basis function are non-overlapping, the individual bands are mutually stochastically orthogonal.

Center of the wavelet defined by eq. (10) is $t^\star = 0$, whereas the radius does not converge. This is a consequence of the slow temporal decay, which property is counterbalanced by the compact support in the frequency region.

The closed-form expression of the moments of statistics of the instantaneous Power Spectral Density Function of the structure response, derived by Basu and Gupta, imposes a restriction to the value of σ to be close unity. The authors use — as a reasonable compromise — the value of $\sigma = 2^{1/4}$. This choice allows to retain the dyadic nature of the wavelet basis. On the other hand, such small value helps to keep fine resolution between individual frequency bands within the whole range.

For the wavelet transform itself Basu and Gupta use values $j = -17, \ldots, 4$ for frequency bands and $\Delta b = 0.02$ for temporal resolution. Such value for Δb corresponds to the sampling frequency of the process under study. Having

$$a_j = \sigma^j \qquad \text{and} \qquad b_i = (i-1)\Delta b \tag{11}$$

we obtain the frequency bands in the form

$$\left(\frac{1}{a_j}\pi, \frac{\sigma}{a_j}\pi\right) \qquad [\text{rad}^{-1}] \tag{12}$$

Table 1 shows frequency ranges for selected values of j enumerated according to equation (12).

Table 1. Frequency ranges for selected components of the decomposition based on the wavelet (10) for $\sigma = 2^{1/4}$

j	a_j	$\omega[\text{rad}^{-1}]$	f[Hz]
$-16.$	0.0625	$(50.265, 59.776)$	$(8., 9.514)$
$-14.$	0.0884	$(35.543, 42.268)$	$(5.657, 6.727)$
$-12.$	0.1250	$(25.133, 29.888)$	$(4., 4.757)$
$-10.$	0.1768	$(17.771, 21.134)$	$(2.828, 3.364)$
$-8.$	0.25	$(12.566, 14.944)$	$(2., 2.378)$
$-6.$	0.3535	$(8.886, 10.567)$	$(1.414, 1.681)$
$-4.$	0.5	$(6.283, 7.472)$	$(1., 1.189)$
$-2.$	0.7071	$(4.443, 5.283)$	$(0.707, 0.841)$
$0.$	1.	$(3.142, 3.736)$	$(0.5, 0.595)$
$2.$	1.4142	$(2.221, 2.642)$	$(0.353, 0.420)$
$4.$	2.	$(1.571, 1.868)$	$(0.25, 0.297)$

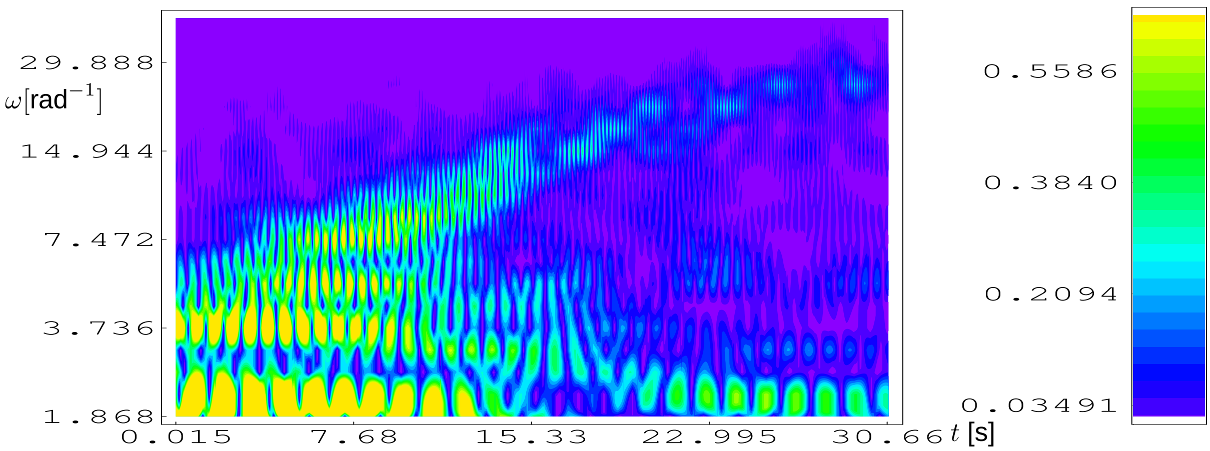

Figure 1. Wavelet-based decomposition of the sine sweep signal (13)

For verification of the procedure we can use the sine sweep example. Decomposition of the artificial signal given by

$$v(t) = \sin(\lfloor 1 + t \rfloor t) \tag{13}$$

for $dt = 0.015$ can be seen on the Figure 1. The bright colors correspond to the large absolute values of the wavelet coefficients.

Examples of decompositions of the selected true seismic records follows on the Figures 2-7.

Table 2. Selected earthquake records

Fig. #	Description
2,3	4 Nov 1973 Ionian earthquake, Lefkada-OTE building station, $M = 5.6$, N-S component
4,5	23 Nov 1973 Azores earthquake, record originates from Portuguese strong-motion network, $M = 4.9$, N-S component
6,7	6 May 1976 Friuli earthquake, Tolmezzo-Diga station, $M = 5.9$, N-S component

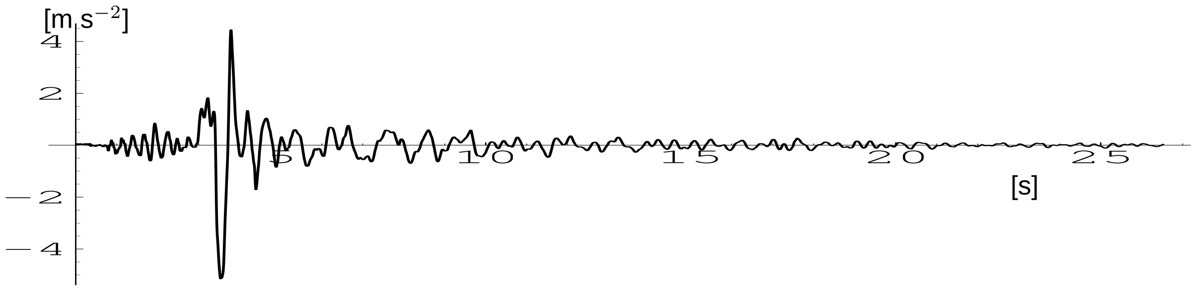

Figure 2. Acceleration time history of the Ionian earthquake, Lefkada-OTE building station, $M = 5.6$, N-S component.

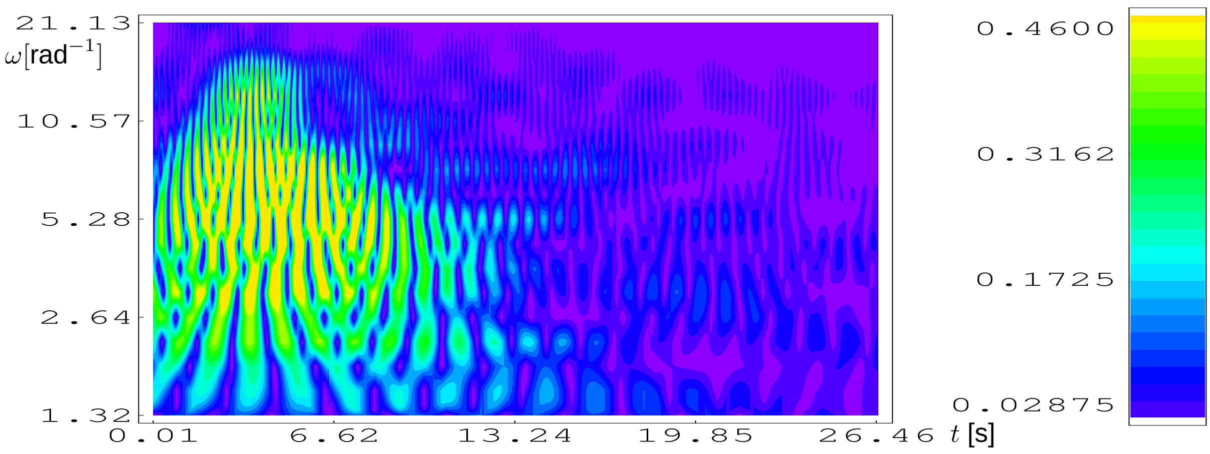

Figure 3. Wavelet based decomposition of the Ionian earthquake accelerogram, Lefkada-OTE building station, $M = 5.6$, N-S component .

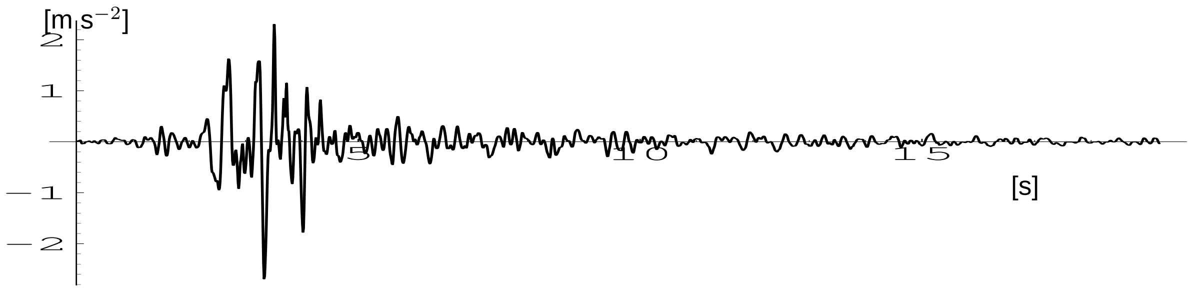

Figure 4. Acceleration time history of the Azores earthquake, record originates from Portuguese strong-motion network, $M = 4.9$, N-S component.

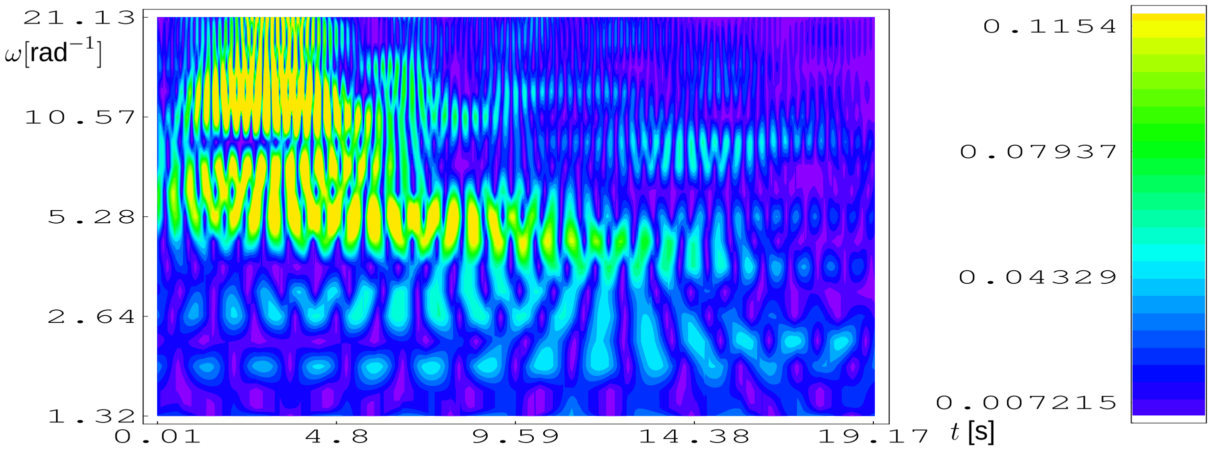

Figure 5. Wavelet based decomposition of the Azores accelerogram (Fig. 4).

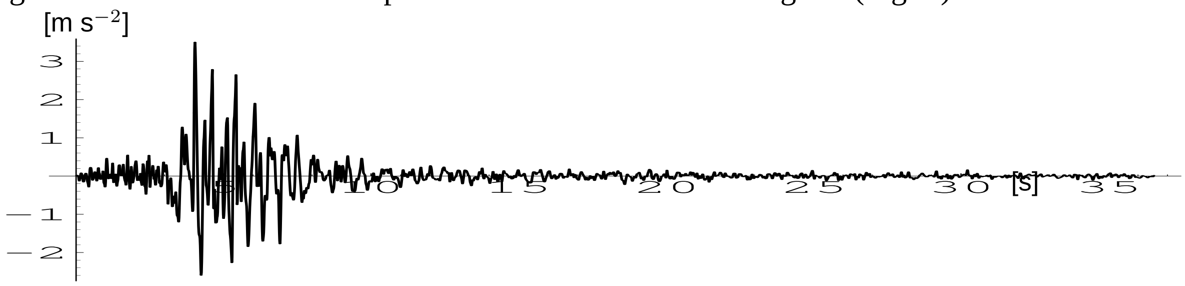

Figure 6. Acceleration time history of the Friuli earthquake, Tolmezzo-Diga station, $M = 5.9$, N-S component

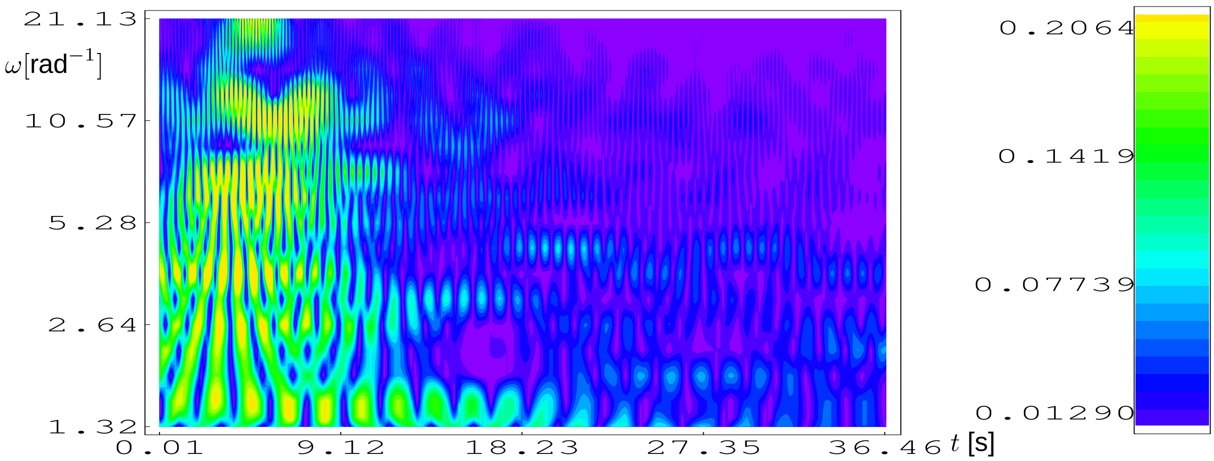

Figure 7. Wavelet based decomposition of the Friuli accelerogram (Fig. 6).

The seismic records were selected from the [Ambraseys et al. 2000] European Strong-Motion Database. The records used are listed in the Table 2.

As can be seen from the images, the areas of significant values of the wavelet coefficients are usually well localized in the time-frequency plane. However, it is not in general an easy task to find a simple 2D function describing approximately the values of the wavelet coefficients.

As an example of the approximating function for the decomposition of the record of the Ionian earthquake (see Fig. 3) can serve the following function

$$\frac{0.0051969\,\omega^2\,\sin\left(\frac{\pi\sqrt{\omega}}{2\sqrt{5}}\right)}{e^{0.000816327\,(-5+t)^2\,\omega^2}} \tag{14}$$

The density plot of the approximating function (14) is depicted in the figure 8.

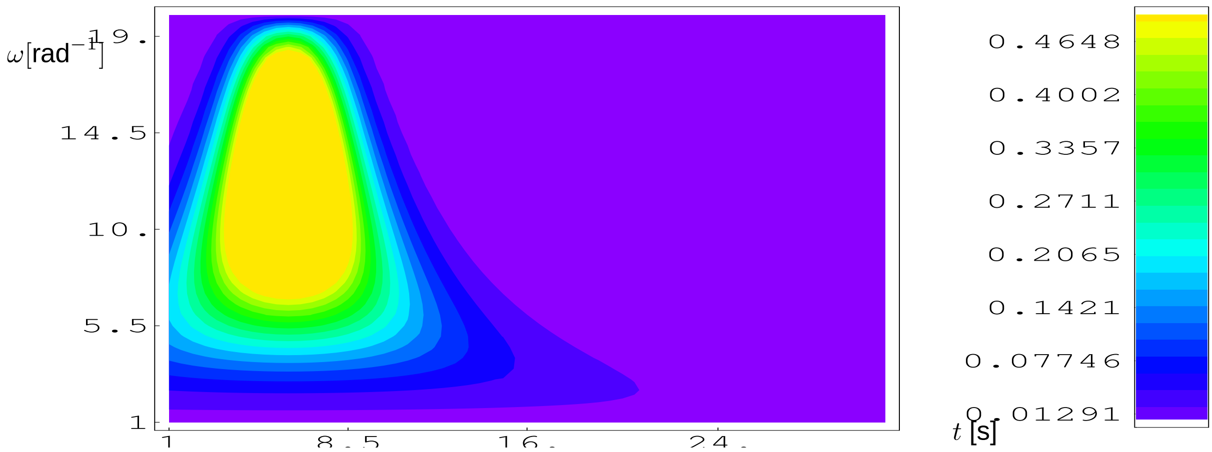

Figure 8. Approximation of the Wavelet based decomposition of the Ionian accelerogram (Fig. 3) via (14).

4. Conclusions

A stochastic formulation of the seismic response of the linear systems based on wavelets requires the wavelet decomposition of the excitation signal as an input data. It is not a very common practice to provide such decomposition for structural engineers. Having a set of typical seismic records one can construct an approximation function, which resemble the distribution of the wavelet coefficients in the time-frequency plane. Special attention should be paid to total energy of the approximated signal.

5. Acknowledgment

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