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APPLICATION OF CONTOUR INTEGRALS TO ASSESSMENT OF NOTCH STABILITY IN LINEAR ELASTICITY

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Abstract: *The integral concepts for calculation of the generalized stress intensity factor and elastic T stress are presented. Both the integrals are based on a validity of the Betti reciprocal theorem. The unique analytical expression of the complementary problem is derived using the Muskhelisvili complex potentials, whereas components of the stress and displacement field are obtained numerically using the Boundary Element Method. The generalized stress intensity factor concept as well as the contour integral for calculation of the elastic T stress are used on single edge notched tension (SENT) specimen to show a validity of the discussed approach. This article yields a comprehensive study of the integral approach for calculation of fracture-mechanics parameters in case when the integration is carried out across the material interface.*

Keywords: Contour integral, generalized stress intensity factor, T stress, material interface.

1 Introduction

The classical linear-elastic fracture mechanics of cracks assumes that the stress singularity at a crack tip equals to 0.5. The stability of a crack inside the body is then assessed using various fracture-mechanics parameters like J integral, stress intensity factor (SIF) K, crack opening displacement (COD), etc. The crack instability occurs when the magnitude of the chosen parameter reaches its critical value that is assumed to be a material characteristic independent on geometry, loading and boundary conditions. The crack stability is recently assessed using the *fracture toughness locus*.

During the past decades, a significant effort has been devoted to the numerical computation of fracture-mechanics parameters. The stress and displacement distribution around the notch tip is generally known as the limit analytical solution. It was shown that for a notch in homogeneous body the stress singularity weakens with increasing notch angle. While for opening mode I the stress singularity exponent continuously decreases from initial value 0.5 for cracks to zero value for smooth edge without defects, the shear mode

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demonstrates a nonsingular behavior for notch angles up to approximately 53° only. From this analysis, the so-called *generalized stress intensity factor* (GSIF) was developed as a counterpart to the classical SIF that is defined only for sharp cracks. It was also shown that the elastic T stress contributes to solution of the given problem only in special cases – for cracks in homogeneous elastic body and for notches and wedges terminating at a bi-material interface [16] – whereas for other configurations the terms with T stress vanish. The easiest way to compute the GSIF is to extrapolate the hoop stresses $\sigma_{\theta\theta}$ at the crack ligament. Although this method is straightforward, it is sometimes difficult to apply such this extrapolation technique on cracks in the vicinity of nonhomogeneous joints and bi-material interfaces, where the significant stress gradient could undermine the extrapolation process as a whole.

This paper presents a powerful integral approach for calculation of GSIF at sharp notches under external loading for pure opening mode I. The method shown here is based on a validity of Betti reciprocal theorem resulting in contour integral, which allows us to separate the GSIF for every mode of loading without the coupling problems known from computation of K_J stress intensity factors using the elastic part of J integral.

The material interface connecting two elastic regions is chosen to show the efficiency and powerfulness of the integral method outlined above. The interface between material 1 and 2 is assumed to be perfectly bonded (e.g. welded), so that the displacements and traction forces are continuous across the interface. The boundary conditions are applied only at boundary nodes; the unknowns at the boundary and inside domains are subsequently obtained using the subdomain Boundary Element Method. The GSIFs for sharp cracks are quantitatively compared with the previous results. Also the variation of T stresses for different crack lengths is discussed in detail.

2 Distribution of stresses and displacements around the V-notches close to material interface

The classical fracture mechanics concepts are based on an assumption that the crack faces are parallel, traction free and the crack tip lies in an homogeneous region. In general case, the stress singularity is of type $1/r^{1-\lambda}$, where λ is eigenvalue of the given problem, and $1-\lambda$ is called *the stress singularity exponent*. For a crack embedded in an elastic region, the exponent fits the value 0.5.

The equilibrium equation of every elastic body without the presence of body forces is governed by biharmonic partial differential equation $\Delta\Delta\Phi = 0$, in which Δ is Laplace's symbol and Φ is an arbitrary function satisfying the given equation and the prescribed boundary conditions. One of the useful functions Φ can be written as follows:

$$\Phi = \sum_{k=1}^{\infty} A_k r^{\lambda_k+1} f_k(\lambda_k, \theta) \quad . \tag{1}$$

Here r and θ are polar coordinates, $f_k(\theta)$ is the angular function, A_k is a constant term of the above expansion, and λ_k is eigenvalue of the given problem that is, for traction free notch faces, dependent only on the notch angle. The only first term of expansion (1) is taken into account to development of ordinary differential equation for angular functions and consequent assessment of its characteristic equation.

The Williams eigenfunction expansion can subsequently be expressed as

$$\sigma_{ij} = \frac{H_{\mathrm{I}}}{\sqrt{2\pi}} r^{\lambda_{\mathrm{I}}-1} f_{ij}^{\mathrm{I}}(\lambda_{\mathrm{I}}, \theta) + \frac{H_{\mathrm{II}}}{\sqrt{2\pi}} r^{\lambda_{\mathrm{II}}-1} f_{ij}^{\mathrm{II}}(\lambda_{\mathrm{II}}, \theta) + T\delta_{1i}\delta_{1j} + \dots \tag{2}$$

in which H_I and H_II are the stress intensity factors for opening and shear mode, respectively. The terms with $\lambda < 1$ are called as *singular*, the term with $\lambda = 1$ is independent on the radial distance and the remainders are *nonsingular* terms. The singular terms with $\lambda = 0.5$ correspond to the so-called *stress intensity factor* that needs to be calculated from numerical analysis of the entire body with prescribed boundary conditions.

The numerical example in this paper will show the difference in computed values of SIF assuming that there is no influence of interface on the calculation, so that the analytical expression for homogeneous body is used.

3 Formulation of contour integrals

3.1 Calculation of the Generalized Stress Intensity Factor

For calculation of the generalized stress intensity factors H_I and H_II (see Eq. (2)), the conservation integrals based on validity of the Betti reciprocal theorem are used here. The integral concept follows from pioneering work by Stern *et al.* [15] and later generalization to notches developed by Sinclair, Okajima and Griffin [12].

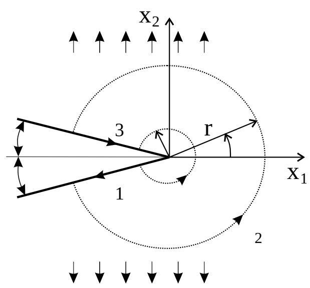

Fig. 1: Segments of the integration path for calculation of the fracture-mechanics parameters of a V-notch

Let we have a closed contour Σ consisting of four segments $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ as shown in Fig. 1. The integral formulation based on the mentioned theorem of work reciprocity [12] is then defined as

$$H_k = \int_\Sigma (\sigma_{ij} u^*_{ik} - \sigma^*_{ijk} u_i) n_j \, \mathrm{d}s \quad , \tag{3}$$

where k denotes the mode of loading (I or II), non-starred components are obtained using the numerical analysis for eigenvalue λ_k and the starred symbols are obtained from analytical solution of the actual problem for eigenvalue $\lambda^*_k = -\lambda_k$. Since we carry out the integration (3) in polar coordinates, i.e. the indexes i, j correspond to r, θ, the only two circular contours contribute to the computation of H_k. Moreover, if we integrate along the short path Γ_4 in limit case when $\varepsilon \to 0$, the identity (3) can be written out in the following form

$$\int_{\Gamma_2} (\sigma_{ij} u^*_{ik} - \sigma^*_{ijk} u_i) n_j \, \mathrm{d}s = \int_{\Gamma_4} (\sigma_{ij} u^*_{ik} - \sigma^*_{ijk} u_i) n_j \, \mathrm{d}s \quad . \tag{4}$$

Substituting all stresses and displacements, the right-hand side of Eq. (4) transforms to the right-hand side of equation

$$\int_{\Gamma_2} (\sigma_{ij} u^*_{ik} - \sigma^*_{ijk} u_i) n_j \, \mathrm{d}s \equiv M_k H_k H^*_k \quad , \tag{5}$$

where the term M_k can be evaluated only using the angular functions as:

$$M_k = \int_{\Gamma_4} \left[f_{ijk}(\lambda_k, \theta) g^*_{ik}(\lambda^*_k, \theta) - f^*_{ijk}(\lambda^*_k, \theta) g_{ik}(\lambda_k, \theta) \right] n_j \, \mathrm{d}\theta \quad . \tag{6}$$

Although, the complementary GSIF H^*_k can be chosen arbitrarily, it is advantageous to use such relation $H^*_k = 1/M_k$. Substituting this equation into (5), the identity for calculation of GSIFs is obtained:

$$H_k = \int_{\Gamma_2} (\sigma_{ij} u^*_{ik} - \sigma^*_{ijk} u_i) n_j \, \mathrm{d}s \quad . \tag{7}$$

Note that, when the contour Γ_2 has a constant radial distance from the notch tip, the integration is carried out only along the radial direction – for $j \equiv r$, because $\vec{n} = (n_r, n_\theta) = (1,0)$.

3.2 Calculation of elastic T-stress for sharp cracks ($2\beta = 0$)

Now let we have two independent states of equilibrium denoted A and B as proposed by Sládek *et al.* [14]. Superscript A will correspond to the given problem with unknown value of T stress, whereas B will address the components referring to a complementary solution satisfying the same boundary conditions as problem A and loaded by a point force parallel with crack faces. Using the superposition principle between these two independent states of equilibrium, the following M integral can be used for subsequent calculation of the T stress:

$$M = \int_{\Gamma_2} \left(\sigma_{ij}^A \varepsilon_{ij}^B n_1 - \sigma_{ij}^A n_j u_{i,1}^B - \sigma_{ij}^B n_j u_{i,1}^A \right) \mathrm{d}\Gamma \quad , \tag{8}$$

where the components denoted by superscript A are obtained from numerical analysis of the given problem and superscript B refers to the analytical solution of the complementary problem (A11), (A13). Here n_j denotes the jth component of the unit outward normal of integration path Γ_2.

Apparently, this integral is nonsingular and therefore its quantification is possible only using classical Gauss quadrature. Moreover this concept is symmetric, so that the integration is required only on one part of the model. The total magnitude of M integral is then obtained considering the given symmetry. For plane strain case, the elastic T stress can be quantified from the known value of M integral (8):

$$M = 2T \left\{ \frac{1-\nu_1^2}{E_1} \frac{c_2}{c_1} A_1 (2\theta_{12} - 2\pi + \sin 2\theta_{12}) - \frac{1-\nu_2^2}{E_2} A_2 (2\theta_{12} + \sin 2\theta_{12}) \right\} \quad . \tag{9}$$

The comprehensive derivation of analytical solution of the complementary field is carried out in Appendix.

Note that in homogeneous case, the identities $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$ are enforced, so that $c_1 = c_2 = 1$, $A_1 = A_2 = -f/(4\pi)$ and the relation between M integral and T stress fits the form

$$M = T f \frac{1-\nu^2}{E} \quad , \tag{10}$$

where f is the arbitrary point force acting at the crack tip. The equation (10) is generally known from previous work [13].

The most essential features of Betti reciprocal theorem

The reciprocal theorem is unaffected by any variation in material characteristics. Therefore also the integration across the material interface is possible. The integral method presented above is useful for calculation of a coefficient corresponding to arbitrary eigenfunction in Williams expansion. As proved by Carpenter [3] or Qian [10], the calculation process is insensitive to eigenvalues of the different order. It was concluded, that there is no ”coupling” in analytical expression of the complementary field. This method is therefore a very efficient tool for calculation of any coefficient corresponding to any eigenfunction.

4 Numerical examples

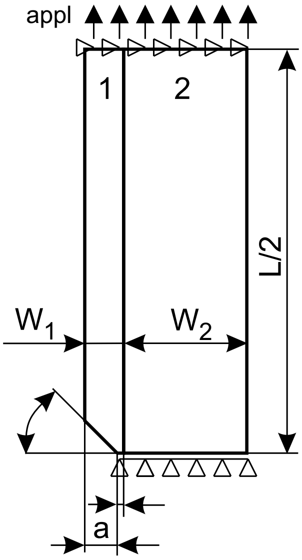

Fig. 2: Model of the interface between two dissimilar materials (unit thickness of the sheet considered)

The SENT specimen was used for numerical calculation of the fracture-mechanics parameters of notches and cracks. Geometry of the test example is defined by constant length $L = 250$ mm and fixed widths $W_1 = 12$ mm, $W_2 = 38$ mm. The external loading $\sigma_{appl} = 100$ MPa is applied onto the upper face of the model, so that the notch is exposed to pure opening mode I. The analysis was performed under the plane strain condition for various notch depths (crack lengths) a. The model geometry and boundary conditions are shown in Fig. 2. The stress and displacement fields were obtained using the subdomain Boundary element method.

To test the integral approach for arbitrary notch angle 2β, the analyses on a single edge notched tension specimen with various notch depths a were carried out. Distribution of the generalized stress intensity factors in terms of a normalized by width of the specimen $W = W_1 + W_2$ for a couple of important notch angles is shown in Fig. 3. The problem was solved for homogeneous case, where $E_1 = E_2 = 2.10^5$ MPa. It is clear, that the GSIF increases with increasing notch depth. For a constant notch depth, the GSIF increases with increasing notch angle. Note that every quantity in Fig. 3 is dimensionally incompatible with each other, and also the critical values vary for different notch angles 2β.

Subsequently, calculation the fracture-mechanics parameters for a crack of various length a was carried out. A crack with the tip in front of the interface (inside domain 1), as well as behind the interface (inside domain 2) was modelled. Note that when $\delta < 0$, the notch tip lies inside domain 1 (in front of the interface), while for $\delta > 0$ the crack had passed through the interface and the crack tip lies inside domain 2.

The contour integral concept outlined in Chapter 3 was used to quantify the fracture-mechanics parameters. The integration path centered at the notch tip was formed by 32 integration points with the trapezoidal integration formula being used for numerical integration.

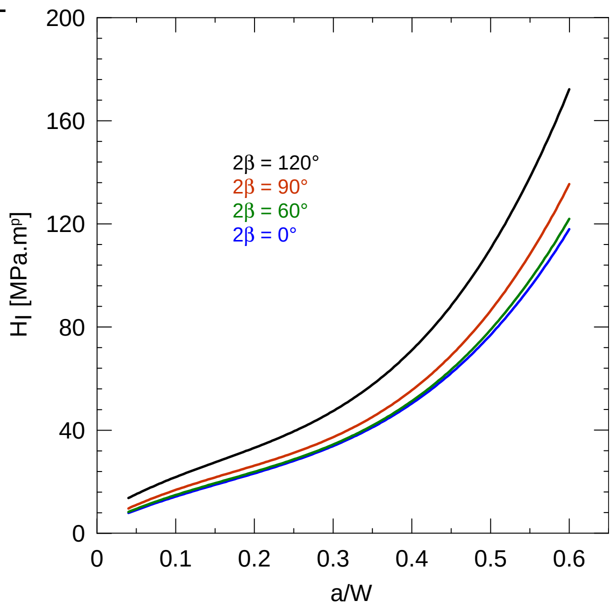

Fig. 3: Variation of GSIF with respect to notch depth a for various notch angles 2β ($E_1 = E_2 = 2.10^5$ MPa)

One of the most advantages of the integral approach over other direct methods is that the same integral identity is used for computation of SIF for sharp cracks and GSIF for notches with arbitrary open angle. A variation of SIF for a crack in front of the interface and behind the interface was calculated. The obtained results show Fig. 4.

It was shown that our recent results are in a good agreement with Menčík [9] and

Atkinson [2] in case when the crack tip is far away from the material interface. In this case, the stress distribution is weakly influenced by the stress gradient along the interface. As the crack propagates within the body toward the interface, the distance $|\delta|$ decreases and the interface plays an essential role in stress redistribution in the vicinity of the crack tip. In this case, however, a significant growth of plastic region could arise, so that the elastic-plastic description of the crack behavior should be assumed. Note that the results for H_I in Fig. 4 were obtained for homogeneous analytical expression of the complementary field (starred symbols in (7). The analytical expression for a crack with the tip removed by δ from the interface has not been used here.

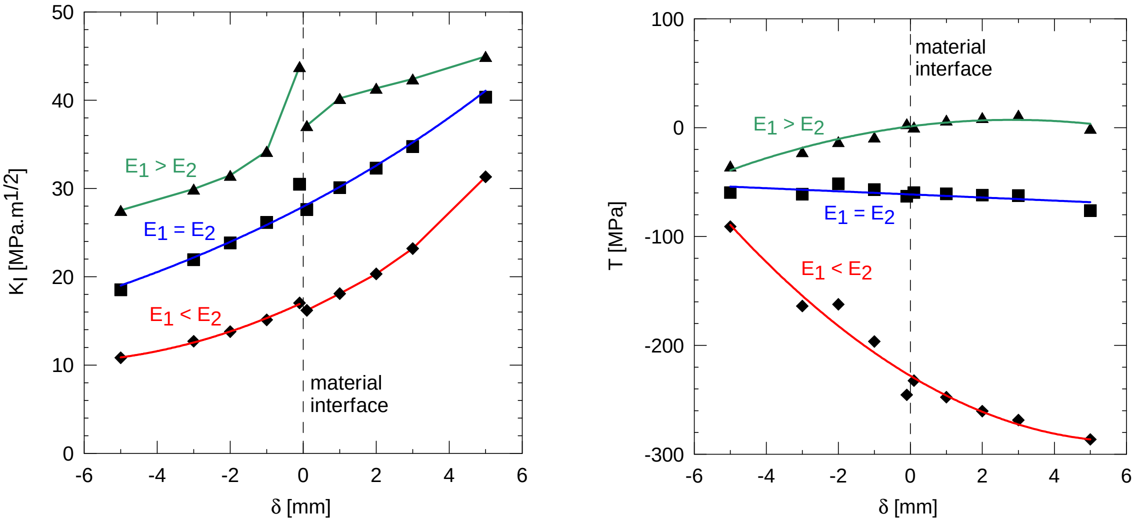

Fig. 4: Influence of ratio E_1/E_2 on the SIF and T stress for sharp cracks. The contour integrals (7) and (9) are used to estimation of the unknown values of K_I and T.

Providing that the crack lies in a homogeneous region or in special cases when the crack tip terminates at the material interface [16], the integral approach for T stress computation can be used. We used this approach to quantify the magnitude of constraint induced by sharp crack tip. For this purpose, the relation between M integral and elastic T stress was derived in Appendix. Also the stress and displacement distribution of the complementary problem transformed into the Cartesian coordinate system was obtained. The computed values of T stresses are presented in Fig. 4.

5 Conclusion

The integral approach for calculation of the generalized stress intensity factor (GSIF) as a counterpart to the commonly used stress intensity factor was presented in Chapter 3. This so-called conservation integrals require the analytical knowledge of the complementary problem, that satisfies the same boundary conditions as the problem being solved. Forasmuch as both the positive and the negative eigenvalue of the same absolute magnitude satisfy the solution of the biharmonic problem $\Delta\Delta\Phi = 0$, the complementary field can be easily obtained from the generally known distribution of stresses and displacements around the notch tip, where the eigenvalue is opposite in sign to the given problem. On the other hand, such this derivation becomes extremely difficult when the crack lies close to material interface.

The similar approach was used to estimate the constraint quantified by T stress. In this case, however, the analytical equation of a crack loaded by a point force is required. The necessary stress and displacement distribution together with the derivatives and relation between M integral and elastic T stress was acquired. This process was based on previous work [14], where the behavior of interface cracks between two dissimilar materials was analyzed.

In comparison to the generally known difference methods of calculation the GSIF and T stress based on extrapolation technique, the use of integral approach seems promising in such cases where the hoop stress gradient undermines accuracy of the solution. Particularly the fracture-mechanics parameters near joints of two dissimilar materials should not be computed using the direct methods, where the influence of the higher order terms of Williams eigenfunction expansion could be essential.

Moreover, the contour integral concept is based on an integration of stresses and displacements far away from the notch tip, where conditions of linear elasticity are satisfied. For a warranted application of such technique on strongly non-homogeneous specimens, an analytical expression of the stress and displacement field should be derived in the future.

6 Acknowledgement

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A Appendix

The integral approach for calculation of elastic T stress requires an analytical solution of the complementary problem – a model with identical geometry as the actual problem, but loaded by a point force f parallel with crack faces. Since use the Muskhelisvili complex potentials, the following relations must be satisfied:

$$\begin{aligned} \sigma_{rr}+\sigma_{\theta\theta} &= 4\mathrm{Re}\{\psi'(\mathrm{z})\} & \text{(A1)}\\ \sigma_{\theta\theta}-\sigma_{rr}+2i\sigma_{r\theta} &= 2e^{2i\theta}[\bar{z}\psi''(z)+\chi''(z)] & \text{(A2)}\\ 2\mu(u_r-iu_\theta) &= e^{i\theta}[\kappa\overline{\psi(z)}-\bar{z}\psi'(z)-\chi'(z)] & \text{(A3)} \end{aligned}$$

where the overline denotes the complex conjugate, $z=re^{i\theta}$, and prime (') is the derivative d/dz. Let we use index a for the components related to *a-th* domain with generally different material properties. Thus we can write the complex potentials ψ and χ as

$$\begin{aligned} \psi_a &= A_a\ln z = A_a(\ln r+i\theta) & \text{(A4)}\\ \chi_a &= B_a z\ln z = B_a re^{i\theta}(\ln r+i\theta) \quad , \end{aligned}$$

where A_a and B_a are the unknown coefficients. Substituting (A4) into Eq. (A1) and (A2), the two following relations define the stress distribution around a crack tip:

$$\sigma_{rr}^{(a)}+\sigma_{\theta\theta}^{(a)}=\frac{4A_a}{r}\cos\theta \quad , \quad \sigma_{\theta\theta}^{(a)}=\frac{A_a+B_a}{r}\cos\theta \quad , \quad \sigma_{r\theta}^{(a)}=\frac{A_a+B_a}{r}\sin\theta \qquad \text{(A5)}$$

Providing that the crack faces are free of tractions, i.e. the crack faces are subjected only to radial stresses σ_{rr}, the conditions $\sigma_{r\theta}(\pm\pi)=\sigma_{\theta\theta}(\pm\pi)=0$ have to be satisfied. From this requirement, also relation $A_a=-B_a$ must be satisfied for both elastic domains a. Now we substitute the obtained relation between A_a and B_a into complex potentials (A4) and subsequently into the following equation:

$$\begin{aligned} 2\mu_a(u_r^{(a)}-iu_\theta^{(a)}) &= e^{i\theta}[\kappa_a\overline{\psi_a(z)}-\bar{z}\psi_a'(z)-\chi_a'(z)] \quad , \quad \text{or:} & \text{(A6)}\\ 2\mu_a(u_r^{(a)}-iu_\theta^{(a)}) &= A_a[(\kappa_a\overline{\ln z}+\ln z)e^{i\theta}+2i\sin\theta] \end{aligned}$$

Now consider the continuous displacements across the interface, so that the real and imaginary parts of relation (A7) must equal. Note that θ_{12} is the angle dependent on δ – the shortest distance between the crack tip and interface, and R_{12} is the distance between the crack tip and the point on integration path that intersects the interface – see Fig. 5. The unique relation between coefficients A_1 and A_2 is then obtained as

$$A_2 = \frac{c_1}{c_2} A_1 \quad , \tag{A7}$$

where the coefficients c_a depend on material properties of the ath domain, angle θ_{12}, and can be computed as follows:

$$c_a = \frac{1}{\mu_a} \left[(\kappa_a + 1) \ln r_{12} \cos\theta_{12} + (\kappa_a - 1)\theta_{12} \sin\theta_{12} \right] \tag{A8}$$

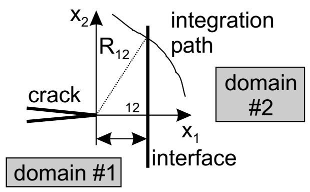

Fig. 5: Properties of interface angle $\theta_{12} = \arccos(\delta/r_{12})$

The unknown value of coefficient A_1 can be obtained using the approach used in [14] as follows:

$$\int_{-\pi}^{-\theta_{12}} \sigma_{rr}^{(1)} e^{i\theta} r d\theta + \int_{-\theta_{12}}^{\theta_{12}} \sigma_{rr}^{(2)} e^{i\theta} r d\theta + \int_{\theta_{12}}^{\pi} \sigma_{rr}^{(1)} e^{i\theta} r d\theta = -f \quad , \tag{A9}$$

where f is a point force applied at the crack tip parallel with the crack flanks. After carrying out the integration in (A9), the coefficient A_1 is obtained in the form

$$A_a = -C^{(a)} \frac{f}{4} \frac{c_1}{c_1(\theta_{12} + \frac{1}{2}\sin 2\theta_{12}) + c_2(\pi - \theta_{12} - \frac{1}{2}\sin 2\theta_{12})} \quad , \tag{A10}$$

where $C^{(1)} = c_2/c_1$ and $C^{(2)} = 1$. Transforming the stresses in Eq. (A5) and displacements obtained using (A7) into Cartesian coordinates, the following set of equations define the stress field:

$$\sigma_{11}^{(a)} = \frac{4A_a}{r} \cos^3\theta \quad , \quad \sigma_{12}^{(a)} = \frac{4A_a}{r} \sin\theta \cos^2\theta \quad , \quad \sigma_{22}^{(a)} = \frac{4A_a}{r} \sin^2\theta \cos\theta \tag{A11}$$

and displacements:

$$\begin{aligned} u_1^{(a)} &= A_a \frac{1+\nu_a}{E_a} \left[(\kappa_a + 1) \ln r + 2\sin^2\theta \right] \\ u_2^{(a)} &= A_a \frac{1+\nu_a}{E_a} \left[(\kappa_a - 1)\theta - \sin 2\theta \right] \end{aligned} \tag{A12}$$

around the crack tip in the vicinity of the material interface. Diferentiating the Cartesian displacements (A12) with respect to the horizontal coordinate, the following two derivatives are obtained:

$$\begin{aligned} u_{1,1}^{(a)} &= A_a \frac{1+\nu_a}{E_a} \frac{1}{r} \left[\kappa_a + 1 - 4\sin^2\theta \right] \cos\theta \\ u_{2,1}^{(a)} &= -A_a \frac{1+\nu_a}{E_a} \frac{1}{r} \left[\kappa_a - 1 - 2\cos 2\theta \right] \sin\theta \end{aligned} \tag{A13}$$

One can easily prove that the integration along the parallel crack faces does not contribute to the overall value of M integral. It means, that the only two contour integrals, one of them faw away from the crack tip and the second very close to the tip are considered. The former integration is carried out numerically, whereas the latter yields the right-hand side of the M integral

$$M^{(A,B)} = 2T \left\{ \frac{1-\nu_1^2}{E_1} \frac{c_2}{c_1} A_1 (2\theta_{12} - 2\pi + \sin 2\theta_{12}) - \frac{1-\nu_2^2}{E_2} A_2 (2\theta_{12} + \sin 2\theta_{12}) \right\} \tag{A14}$$

Note that (A14) is useful only for plane strain case. Nevertheless the identical approach can be used also to derive the final relation between M integral and T stress for plane stress case.

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