National Conference with International Participation
ENGINEERING MECHANICS 2001
Svratka, Czech Republic, May 14 - 17, 2001

INFLUENCE OF PERIODICALLY CHANGING PARAMETER OF GEAR MESH TO DYNAMIC BEHAVIOUR AND STABILITY OF CAR GEARBOX

Jan DUPAL, Tomáš KURUC •

Summary: *The paper deals with a way of mathematical simulation of dynamic behaviour of car gear box model. For mathematical model assembling of rotors, differential and gearbox the 1D finite elements, rigid and shell finite elements are used respectively. The problems of dynamic stability are solved by means of Floquet's theory.*

This paper was written as a result of research project MSM 235200003

1. INTRODUCTION

A car gearbox can be modelled as a system consisting of several subsystems such as two rotors with speed gear wheels, differential and gearbox. These isolated subsystems can be represented by linear mathematical model with constant mass, gyroscopic+damping and stiffness matrices. For whole assembled mathematical model a modal synthesis method [1] can be used respecting non-linear or time dependent couplings connecting individual linear subsystems. The goal of this paper is to show a way of modelling of car gearbox containing the time dependent linear couplings in gear meshes.

2. COMPONENTS OF PHYSICAL MODEL

A scheme of a car gearbox part is depicted in fig. 1

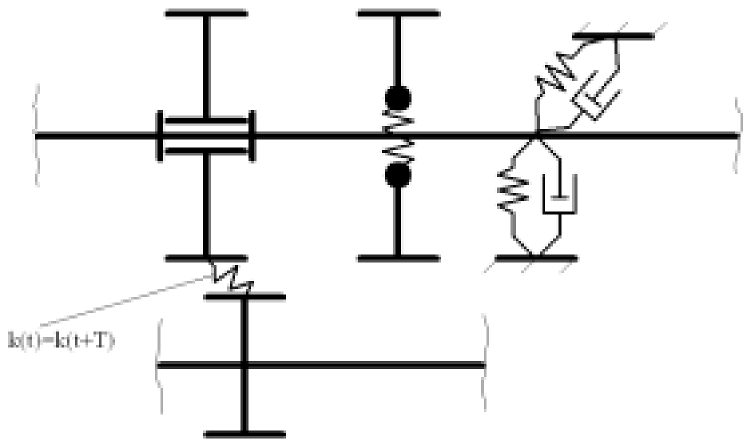


Fig.1

The gearbox components can be modelled by three groups of elements:

• Doc. Dr. Ing. Jan Dupal, Tomáš Kuruc, Dept. of. Mech., University of West Bohemia in Pilsen, Univerzitní 22, 306 14 Plzeň, dupal@kme.zcu.cz, kury@students.zcu..cz

a) rotors – 1D finite rotor elements, rigid wheels, massless springs and dampers (software of Dept. of Mech.)

b) differential – combination of rigid body and flexible rotor elements (software of Dept. of Mech.)

c) gearbox – shell and 3D elements (FEM professional software ANSYS)

Corresponding mathematical model has a form

$$\mathbf{M}\ddot{\mathbf{q}}(t)+[\mathbf{B}_i+\mathbf{B}_c(t)]\dot{\mathbf{q}}(t)+[\mathbf{K}_i+\mathbf{K}_c(t)]\mathbf{q}(t)=\hat{\mathbf{f}}(t) \tag{1}$$

where

$$\mathbf{M}=\begin{bmatrix}\mathbf{M_1} & & & \\ & \mathbf{M_2} & & \\ & & \mathbf{M_D} & \\ & & & \mathbf{M_G}\end{bmatrix},\ \mathbf{B}_i=\begin{bmatrix}\mathbf{B_1} & & & \\ & \mathbf{B_2} & & \\ & & \mathbf{B_D} & \\ & & & \mathbf{B_G}\end{bmatrix},$$

$$\mathbf{K}_i=\begin{bmatrix}\mathbf{K_1} & & & \\ & \mathbf{K_2} & & \\ & & \mathbf{K_D} & \\ & & & \mathbf{K_G}\end{bmatrix}, \tag{2}$$

are mass, damping+gyroscopic and stiffness matrices of each other isolated subsystems respectively. $\mathbf{B}_c(t)=\mathbf{B}_0+\hat{\mathbf{B}}(t)$, $\mathbf{K}_c(t)=\mathbf{K}_0+\hat{\mathbf{K}}(t)$ are time dependent damping and stiffness coupling matrices respectively.

Having performed a modal analysis of the individual isolated subsystems respecting Kelvin-Voigth damping [2] we can rewrite (1) into form

$$\ddot{\mathbf{x}}(t)+[\mathbf{\Gamma}+\mathbf{V}^{\mathrm{T}}\mathbf{B}_c(t)\mathbf{V}]\dot{\mathbf{x}}(t)+[\mathbf{\Lambda}+\mathbf{V}^{\mathrm{T}}\mathbf{K}_c(t)\mathbf{V}]\mathbf{x}(t)=\mathbf{V}^{\mathrm{T}}\hat{\mathbf{f}}(t), \tag{3}$$

where $\mathbf{V}$ and $\mathbf{\Lambda}$ are modal and spectral matrices of the each other isolated subsystems

$$\mathbf{\Lambda}=\begin{bmatrix}\mathbf{\Lambda}_1 & & & \\ & \mathbf{\Lambda}_2 & & \\ & & \mathbf{\Lambda}_D & \\ & & & \mathbf{\Lambda}_G\end{bmatrix},\ \mathbf{V}=\begin{bmatrix}\mathbf{V}_1 & & & \\ & \mathbf{V}_2 & & \\ & & \mathbf{V}_D & \\ & & & \mathbf{V}_G\end{bmatrix},\ \mathbf{\Gamma}=\mathbf{V}^T\mathbf{B}_i\mathbf{V} \tag{4}$$

where number of columns of $\mathbf{V}\in\mathbf{R}^{n,m}$ is equal to sum of the respected eigenvectors of individual subsystems. $\mathbf{\Gamma}$ is diagonal matrix because of proportionality. Having added a trivial identity

$$\dot{\mathbf{x}}(t)-\dot{\mathbf{x}}(t)=\mathbf{0} \tag{5}$$

to (3) we can rewrite both equation into compact form

$$\mathbf{N}\dot{\mathbf{u}}(t)-\mathbf{P}(t)\mathbf{u}(t)=\mathbf{f}(t) \tag{6}$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} = \mathbf{N}^{-1}, \ \mathbf{P}(t) = \begin{bmatrix} -\mathbf{\Lambda} - \mathbf{V}^{\mathrm{T}}\mathbf{K}_c(t)\mathbf{V}, & -\mathbf{\Gamma} - \mathbf{V}^{\mathrm{T}}\mathbf{B}_c(t)\mathbf{V} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{7}$$

and

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} \mathbf{V}^T \hat{\mathbf{f}}(t) \\ \mathbf{0} \end{bmatrix}. \tag{8}$$

Eq. (6) can be further rearranged into form

$$\dot{\mathbf{u}}(t) = \mathbf{A}(t)\mathbf{u}(t) + \mathbf{b}(t) \tag{9}$$

where

$$\mathbf{A}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Lambda} - \mathbf{V}^{\mathrm{T}}\mathbf{K}_c(t)\mathbf{V}, & -\mathbf{\Gamma} - \mathbf{V}^{\mathrm{T}}\mathbf{B}_c(t)\mathbf{V} \end{bmatrix}, \mathbf{f}(t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{V}^T \hat{\mathbf{f}}(t) \end{bmatrix}. \tag{10}$$

Matrix $\mathbf{A}(t) \in \mathbf{R}^{2m,2m}$ is time periodic and has a period T. For stability recognition we use Floquet's theory. It means to solve homogeneous solution of (10)

$$\dot{\mathbf{u}}(t) = \mathbf{A}(t)\mathbf{u}(t) \tag{11}$$

with independent initial conditions. Let us suppose the existence of $2m$ linearly independent solutions (fundamental set of solutions) $\mathbf{x}_i(t), \ i = 1, 2, ..., 2m.$ A fundamental matrix solution can be introduced in the form of:

$$\mathbf{U}(t) = [\mathbf{u}_1(t), \mathbf{u}_2(t), ..., \ \mathbf{u}_{2m}(t)], \tag{12}$$

which satisfies a condition

$$\dot{\mathbf{U}}(t) = \mathbf{A}(t)\mathbf{U}(t). \tag{13}$$

Time variable t can be replaced by $\tau = t + T$ and (13) has a form

$$\frac{d\mathbf{U}}{d\tau} = \mathbf{A}(\tau - T)\mathbf{U} = \mathbf{A}(\tau)\mathbf{U}, \tag{14}$$

because of periodicity of $\mathbf{A}(t)$. Hence if

$$\mathbf{U}(t + T) = \mathbf{U}(t)\mathbf{Z}, \quad \mathbf{Z} \in \mathbf{R}^{2m,2m}. \tag{15}$$

If initial fundamental matrix has a form

$$\mathbf{U}(0) = \mathbf{I}, \tag{16}$$

where $\mathbf{I} \in \mathbf{R}^{2m,2m}$ is identity matrix, then

$$\mathbf{Z} = \mathbf{U}(T) \tag{17}$$

is so called monodromy matrix, whose eigenvalues decide about stability of the system. If all eigenvalues lie in unit circle in Gauss plane, the system is stable otherwise the system is unstable.

Proof:

Having passed Jordan canonical transformation of $\mathbf{Z}$

$$\mathbf{P}^{-1}\mathbf{Z}\mathbf{P} = \mathbf{J}, \tag{18}$$

where $\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{2m}]$ is right eigenvector matrix of the $\mathbf{Z}$ and $\mathbf{J}$ is Jordan's matrix we can use a canonical transformation of the fundamental matrix

$$\mathbf{X}(t) = \mathbf{V}(t)\mathbf{P}^{-1}. \tag{19}$$

From (19) follows

$$\mathbf{X}(t+T) = \mathbf{V}(t+T)\mathbf{P}^{-1} \tag{20}$$

and the inverse relation

$$\mathbf{V}(t+T) = \mathbf{X}(t+T)\mathbf{P}. \tag{21}$$

Respecting (19) relation (15) can be rewritten as

$$\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{Z} = \mathbf{V}(t)\mathbf{P}^{-1}\mathbf{Z}. \tag{22}$$

Substituting (22) to (21) we can write

$$\mathbf{V}(t+T) = \mathbf{V}(t)\mathbf{P}^{-1}\mathbf{Z}\mathbf{P} = \mathbf{V}(t)\mathbf{J}. \tag{23}$$

In case $\mathbf{J}$ is diagonal matrix containing eigenvalues of monodromy matrix λ_i on the diagonal we can write relation for modal coordinates $\mathbf{v}_i$ in N - multiple of period T in the form

$$\mathbf{v}_i(t+T) = \lambda_i \mathbf{v}_i(t), \quad \mathbf{v}_i(t+NT) = \lambda_i^N \mathbf{v}_i(t). \tag{24}$$

From the last relation follows the condition if $|\lambda_i| < 1,$ system is stable because

$$\lim_{N\to\infty} \mathbf{v}_i(t+NT) = \lim_{N\to\infty} \lambda_i^N \mathbf{v}_i(t) = \mathbf{0} \tag{25}$$

and vice versa. In case $\mathbf{J}$ is nondiagonal for example

$$\mathbf{J} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & 1 & \\ & & \lambda_3 & \\ & & & \ddots \end{bmatrix} \tag{26}$$

follows

$$\mathbf{v}_i(t+NT) = \lambda_i^N \mathbf{v}_i(t) \quad \text{for } i \neq 3 \tag{27}$$

and

$$\mathbf{v}_i(t+NT)=\sum_{j=0}^{N-1}\lambda_2^j\lambda_3^{N-j-1}\mathbf{v}_2(t)+\lambda_3^N\mathbf{v}_3(t) \quad \text{for } i=3 \tag{28}$$

In both cases if $|\lambda_i|<1,$ the system is stable. The proof can be performed for larger Jordan's cell in the similar way.

3. NUMERICAL VERIFICATION

Let us find out the angular speed unstable region of simple exhibition system depicted in fig. 2

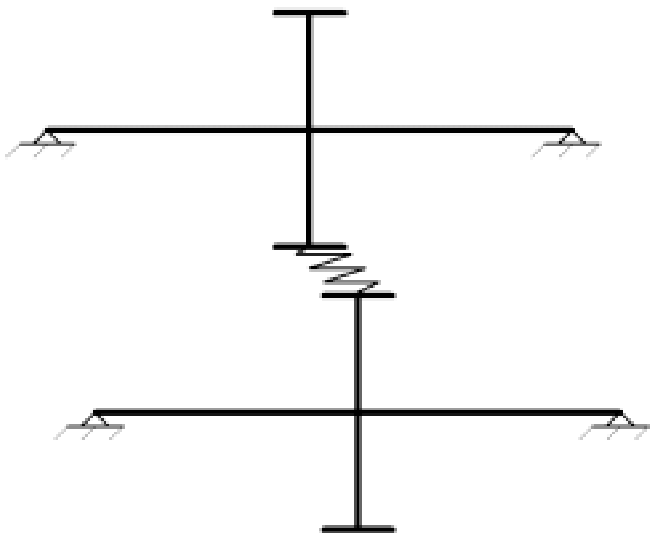


Fig. 2

Let suppose the upper driving shaft has angular speed ω_1 and z is the driving wheel number of teeth. The angular speed of periodical change of tooth stiffness and corresponding period can be obtained in form

$$\omega = z\omega_1, \quad T=\frac{2\pi}{\omega}=\frac{2\pi}{z\omega_1}. \tag{29}$$

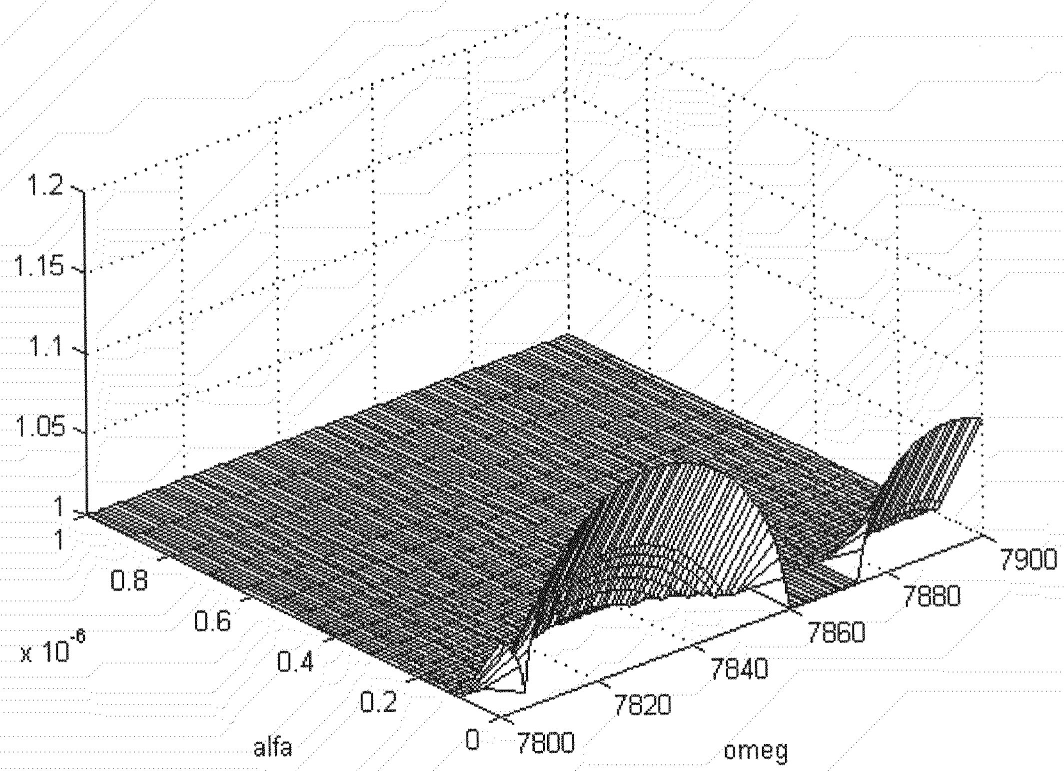
The first ten eigenfrequencies of the system with central stiffness of teeth are as follows

i	1	2	3	4	5
$\Omega_i\left[rad/s.10^4\right]$	0	0.0728	0.0801	0.1613	0.1737
i	6	7	8	9	10
$\Omega_i\left[rad/s.10^4\right]$	0.3504	0.3969	0.7847	0.8593	1.8226

The most sensitive eigenfrequency to the tooth stiffness modulation is Ω_8. Let us change modulation frequency in the region $\omega\in\langle 7800,7900\rangle[rad/s]$.
Corresponding region of revolutions is

$$n\in\langle 3724.2,3592.3\rangle[rev/\min], \qquad n=\frac{\omega_{rev}}{2\pi}60=\frac{30\omega}{\pi\, z}. \tag{30}$$

The measure of instability dependence (modulus of maximal eigenvalue of monodromy matrix) on ω and proportional damping coefficient β ($\mathbf{B} = \alpha\mathbf{M} + \beta\mathbf{K},\ (\alpha = 0)$) is depicted in fig. 3



This academic example shows that the parametric instability can occur from the result of small damping and high stiffness modulation (50%). However maximal modulation of real gearboxes is about 15%. Simulation of dynamic behaviour of real car gearbox was performed but it is not allowed to publish any results.

4. Conclusion

The calculations of real car gearboxes proved that tooth stiffness time periodicity can be neglected. Taking a central tooth stiffness into account we can achieve a very correct results in case of low modulation .

5. References

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2. Zapoměl, J.: The approaches to material damping implementing to computer models of mechanical systems. Thesis to degree of as. Prof., Ostrava, 1998. (In Czech)